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About One Problem for the MAC Model for the Heat Conduction Equation

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Abstract

The aim of this work is to study a class of problems for continuum models in mechanics. We construct the MAC model for the parabolic equation which could include the physical solution appying Newton's law of cooling. A notion of a generalized solution is introduced. We apply Galerkin method to prove the existence of a generalized solution. The proof of uniqueness of a generalized solution is based on the obtained energy inequality.

Keywords: Parabolic equations; MAC model; apriori estimate; loaded equations

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1 Introduction

To introduce the mathematical problem, consider the class of problems arising in mechanics [1].

Many continuum models in mechanics include contradictions in case of applied point boundary conditions. The method of additional conditions or MAC can be applied to the boundary value problems of mathematical physics in case if a classical solution does not exist or a nonphysical solution is obtained. This method allows to transform the obtained nonphysical solution to the physically acceptable form. The MAC was introduced in the scheme of Dugdale-Barrenblat [2,3] in fracture mechanics. This scheme was applied to the linear elastic crack problem in which the linear elastic solution has singularity near the tip of a crack. To avoid this singularity Dugdale and Barrenblat introduced additional yield stresses near the tip. The applied nonsingular condition gave the size of the zone, where the stresses are applied. This new condition gave the value of the applied additional stresses, which are six times more then the given stresses at infinity. The stress concentration factor corresponds to the experiments of Griffith's and Inglis [4]. The MAC scheme was developed in [5], where the MAC solution for the Laplace equation in an angle was considered.

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One of the possible MAC problems can be introduced as follows. Consider a body and take some control volume which includes a fixed number of particles. The control volume is surrounded by a control surface. The particles which are inside the control surface are called internal particles and they belong to the control volume. The particles which are outside the control volume are the external particles and they do not belong to the control volume. All other particles belong to the boundary particles of the control volume. There are interactions between particles for example, according to Newtons law of cooling; the resultant of interactions applied to all internal particles of the control volume from the external particles is the internal body flux. The interactions applied to the boundary particles of the control volume from the external particles are the surface fluxes. The Fourier law could be accepted for the internal surface heat fluxes and the Newtons cooling law is taken to describe the nonlocal body heat fluxes.

Consider the classical heat conduction problem with the equation according to [6]

$$k\Delta u - q_0 + q_1 = c_0 \rho u_t, \tag{1.1}$$

where ρ is the mass-density of the body per unit volume, c_0 is the specific heat, k is the coefficient of thermal conduction, q_0 is a rate of internal body heat flux per unit volume, q_1 is a rate of internal heat generation per unit volume produced in the body. The introduced in (1.1) term q_0 can be taken using the Newtons law of cooling in the form

$$q_0 = \frac{1}{V}\int_V \alpha \left(u(x,t) - u(\xi,t)\right)\, d\xi,$$

where V is the volume of the body, α is a constant in Newtons law of cooling which can depend in general on the coordinates of the body. The correspondent initial and boundary conditions should be added to create a well posed initial boundary value problem. It was shown in [7] that heat problems have nonphysical solution in case of a given point boundary condition.

In mathematical literature the equations of such type are called parabolic integro-differential or loaded equations. Various classes of loaded equations were studied in [8]. Note here some recent works dealing with parabolic integro-differential equations [9,10]. See also references therein.

Motivate by this, we consider a new MAC model for a parabolic equation in the form of integro-differential equation which could include the physical solution into consideration.

2 Preliminaries

In the domain $Q_T = \{(x,t)\colon 0 < x < l,\ 0 < t < T\}$ consider the equation

$$u_t - (a(x,t)u_x)_x + u(x,t) - \frac{1}{l}\int_0^l u(s,t)\, ds = f(x,t) \tag{2.1}$$

subject to the initial condition

$$u(x,0) = \varphi(x), \quad x \in (0,l) \tag{2.2}$$

and the boundary conditions

$$u(0,t) = 0, \quad t \in (0,T), \tag{2.3}$$

$$u(l,t) = 0, \quad t \in (0,T). \tag{2.4}$$

Denote

$$\Phi = \frac{1}{l}\int_0^l u(s,t)\, ds.$$

Let $W_2^1(Q_T)$, $W_2^{1,0}(Q_T)$ be the usual Sobolev spaces. We shall define

$$V_2 = \left\{ u\colon u \in W_2^{1,0},\ ||u||^2 = |u|^2 = ess \sup_{0\le t\le T}\int_\Omega u^2\, dx + \int_{Q_T} u_x^2\, dx\, dt \right\}$$

$$\widehat{W}_2^1(Q_T) = \{u\colon u \in W_2^1(Q_T),\ u(x,T) = 0\}$$

First we introduce the notion of a generalized solution of the problem (2.1)–(2.4) using the standart method [(10)]. Given that $u(x,t)$ is a classical solution of the problem (2.1)–(2.4) we multiply both sides of the identity (2.1) by some function $\eta(x,t) \in \widehat{W}_2^1(Q_T)$, $\eta(0,t) = \eta(l,t) = 0$ and integrate the obtained equality over Q_T

$$\int_{Q_T} (u_t - (au_x)_x + u - \Phi)\eta(x,t)\,dx\,dt = \int_{Q_T} f(x,t)\eta(x,t)\,dx\,dt$$

It follows from integration by parts in the first and the second terms, the conditions (2.2)–(2.4) and the properties of the function $\eta(x,t)$ that

$$\int_{Q_T} (-u\eta_t + au_x\eta_x + u\eta - \Phi\eta)\,dx\,dt = \int_{Q_T} f(x,t)\eta(x,t)\,dx\,dt + \int_0^l \varphi(x)\eta(x,0)\,dx \tag{2.5}$$

Definition 2.1. We say that a function $u(x,t) \in V_2(Q_T)$ is a generalized solution of the problem (2.1)–(2.4) provided for any function $\eta \in \widehat{W}_2^1(Q_T)$, $\eta|_{S_T} = 0$ the function $u(x,t)$ satisfies the integral identity (2.5).

3 Main Results

The main result of this work is the following statement.

Theorem 3.1. *Let $f(x,t) \in L_{2,1}(Q_T)$, $\varphi(x) \in L_2(0,l)$, $a(x,t) \in C(\overline{Q}_T)$, $\frac{\nu}{2} \le a(x,t) \le \mu$. Then there exists a unique generalized solution of the problem (2.1)–(2.4).*

Proof. The proof of the thorem is organized as follows. In the first part we obtain the energy inequality from which uniqueness of the solution of the problem (2.1)–(2.4) follows immediately. In the second part applyng the Galerkin method we construct approximations to the generalized solution and use apriori estimates to provide convergence of approximations. After that we justify that a limit of approximations is the generalized solution of the (2.1)–(2.4).

1. Energy inequality. Let $u(x,t) \in W_2^1(Q_T)$ and satisfy the integral identity (2.5) for any function $\eta \in \widehat{W}_2^1(Q_T)$, $\eta|_{Q_T} = 0$. We take

$$\eta(x,t) = \begin{cases} u(x,t), & 0 < t < \tau, \\ 0, & \tau \le t < T, \end{cases} \qquad \text{where} \quad 0 < \tau < T.$$

After inegration by parts in the first term in (2.5) with the chosen η we get

$$\int_0^\tau \int_0^l au_x^2\,dx\,dt + \int_0^\tau \int_0^l u^2\,dx\,dt + \frac{1}{2}\int_0^l u^2(x,\tau)\,dx - \int_0^\tau \int_0^l \Phi u\,dx\,dt =$$
$$= \frac{1}{2}\int_0^l \varphi^2(x)\,dx + \int_0^\tau \int_0^l fu\,dx\,dt.$$

Hence,

$$\frac{1}{2}\int_0^l u^2(x,\tau)\,dx + \int_0^\tau \int_0^l au_x^2\,dx\,dt = \frac{1}{2}\int_0^l \varphi^2(x)\,dx + \int_0^\tau \int_0^l fu\,dx\,dt -$$
$$- \int_0^\tau \int_0^l u^2\,dx\,dt + \int_0^\tau \int_0^l \Phi u\,dx\,dt. \tag{3.1}$$

Therefore, for the solution of the problem (2.1)–(2.4) $u(x,t) \in W_2^1(Q_T)$ the integral identity (3.1) is valid.

Now, consider a functional sequence $u^m \in W_2^1(Q_T)$ which satisies (2.5) with the chosen η and hence, it satisfies (3.1).

The set $W_2^1(Q_T)$ is dense in $V_2^{1,0}$ and hence, it is dense in $V_2(Q_T)$. Thus there exists a function $u^* \in V_2(Q_T)$ such that $\left\{ \begin{array}{l} u^m \to v^* \\ u_x^m \to v_x^* \end{array} \right.$ strongly in $L_2(Q_T)$ as $m \to \infty$. This implies that $u^m \to u^*$ strongly in $L_2(Q_T)$ as $m \to \infty$. Hence, letting $m \to \infty$ in the identity (3.1) for the functions u^m we see that (3.1) is also valid for the function $v^* \in V_2(Q_T)$.

Therefore, if the function $u \in V_2(Q_T)$ is a generalized solution of the problem (2.1)–(2.4) then it satisfies the integral identity (3.1).

Now, we obtain the apriori estimate of the solution of the problem (2.1)–(2.4). Let $u(x,t)$ be the solution of the problem (2.1)–(2.4) and hence, the identity (3.1) holds.

$$\frac{1}{2}\int_0^l u^2(x,\tau)\,dx + \int_0^\tau\int_0^l a u_x^2\,dx\,dt = \frac{1}{2}\int_0^l \varphi^2(x)\,dx + \int_0^\tau\int_0^l fu\,dx\,dt -$$

$$-\int_0^\tau\int_0^l u^2\,dx\,dt + \int_0^\tau\int_0^l \Phi u\,dx\,dt, \quad \text{where} \quad 0 < \tau < T.$$

First estimate the second term in the right-hand side of (3.1).

$$\left|\int_0^\tau\int_\Omega f(x,t)u(x,t)\,dx\,dt\right| \le \int_0^\tau \left(\int_\Omega f^2\,dx\right)^{1/2}\left(\int_\Omega u^2\,dx\right)^{1/2} dt \le$$

$$\le \max_{0\le t\le\tau}\|u\|_{L_2(\Omega)}\|f\|_{2,1,Q_\tau}. \tag{3.2}$$

Second we estimate the fourth term in the right-hand side of (3.1) using the inequality $|ab| \le \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$ and the Cauchy inequality as follows.

$$\left|\int_0^\tau\int_0^l \Phi u\,dx\,dt\right| = \left|\frac{1}{l}\int_0^\tau\int_0^l u\int_0^l u(s,t)\,ds\,dx\,dt\right| \le$$

$$\le \frac{1}{2l}\int_0^\tau\int_0^l u^2(x,t)\,dx\,dt + \frac{1}{2l}\int_0^\tau\int_0^l\left(\int_0^l u(s,t)\,ds\right)^2 dx\,dt \le$$

$$\le \frac{1}{2l}\int_0^\tau\int_0^l u^2(x,t)\,dx\,dt + \frac{l}{2}\int_0^\tau\int_0^l u^2\,dx\,dt \tag{3.3}$$

Therefore from (3.1), (3.2), (3) it follows that

$$\frac{1}{2}\int_0^l u^2(x,\tau)\,dx + \int_0^\tau\int_0^l \frac{\nu}{2}u_x^2\,dx\,dt \le \frac{1}{2}\int_0^l \varphi^2(x)\,dx + \max_{0\le t\le\tau}\|u\|_{L_2(\Omega)}\|f\|_{2,1,Q_\tau} +$$

$$+\int_0^\tau\int_0^l u^2\,dx\,dt + \frac{1}{2l}\int_0^\tau\int_0^l u^2(x,t)\,dx\,dt + \frac{l}{2}\int_0^\tau\int_0^l u^2\,dx\,dt$$

And hence,

$$\int_0^l u^2(x,\tau)\,dx + \nu\int_0^\tau\int_0^l u_x^2\,dx\,dt \le \int_0^l \varphi^2(x)\,dx + 2\max_{0\le t\le\tau}\|u\|_{L_2(\Omega)}\|f\|_{2,1,Q_\tau} +$$

$$+ C\int_0^\tau\int_0^l u^2\,dx\,dt, \quad \text{where} \quad C = 2\left(1 + \frac{1}{2l} + \frac{l}{2}\right) \tag{3.4}$$

Follwing [11] we define $y(\tau) = \max_{0\le t\le\tau}\|u\|_{L_2(\Omega)}$. Then

$$\int_0^\tau\int_0^l u^2\,dx\,dt = \int_0^\tau \|u\|^2_{L_2(0,l)}\,dt \le \tau y^2(\tau), \tag{3.5}$$

$$\|\varphi\|^2_{L_2(0,l)} = \int_0^l \varphi^2\, dx = \int_0^l u^2(x,0)\, dx = \left(\int_0^l u^2(x,0)\, dx\right)^{1/2} \times$$
$$\times \left(\int_0^l u^2(x,0)\, dx\right)^{1/2} \le \max_{0\le t\le \tau} \|u\|_{L_2(0,l)} \|u(x,0)\|_{L_2(0,l)} =$$
$$= y(\tau)\|\varphi\|_{L_2(0,l)}. \tag{3.6}$$

Using (3.4), (3.5), (3.6) we obtain

$$\int_0^l u^2(x,\tau)\, dx + 2\nu \int_0^\tau \int_0^l u_x^2\, dx\, dt \le y(\tau)\left(\|\varphi\|_{L_2(0,l)} + 2\|f\|_{2,1,Q_\tau}\right) +$$
$$+ C\tau y^2(\tau). \tag{3.7}$$

This implies that

$$\|u\|^2_{L_2(0,l)} \le y(\tau)\left(\|\varphi\|_{L_2(0,l)} + 2\|f\|_{2,1,Q_\tau}\right) + C\tau y^2(\tau).$$

Hence,

$$y^2(\tau) \le y(\tau)(\|\varphi\|_{L_2(0,l)} + 2\|f\|_{2,1,Q_\tau}) + C\tau y^2(\tau) \tag{3.8}$$

And

$$y(\tau) \le \sqrt{y(\tau)}\left(\|\varphi\|_{L_2(\Omega)} + 2\|f\|_{2,1,Q_\tau}\right)^{1/2} + \sqrt{C\tau}y(\tau). \tag{3.9}$$

Also the inequality (3.7) implies that

$$\|u_x\| \le \sqrt{y(\tau)}\left(\|\varphi\|_{L_2(0,l)} + 2\|f\|_{2,1,Q_\tau}\right)^{1/2}\nu^{-1/2} +$$
$$+ \nu^{-1/2}\sqrt{C\tau}y(\tau). \tag{3.10}$$

Therefore, the estimates (3.9), (3.10) give us

$$|u|_{Q_\tau} \le (1+\nu^{-1/2})\left(\|\varphi\|_{L_2(0,l)} + 2\|f\|_{2,1,Q_\tau}\right)^{1/2}|u|^{1/2}_{Q_\tau} +$$
$$+(1+\nu^{-1/2})\sqrt{C\tau}|u|_{Q_\tau}.$$

Then for $\tau < \tau_1 \equiv \dfrac{\nu}{C(\sqrt{\nu}+1)^2}$ we have

$$\frac{\sqrt{\nu} - (1+\sqrt{\nu})\sqrt{C\tau}}{\sqrt{\nu}}|u|^{1/2}_{Q_\tau} \le \frac{\sqrt{\nu}+1}{\sqrt{\nu}}\left(\|\varphi\|_{L_2(0,l)} + 2\|f\|_{2,1,Q_\tau}\right)^{1/2}.$$

And hence,

$$|u|_{Q_\tau} \le \frac{(1+\sqrt{\nu})^2}{(\sqrt{\nu} - (1+\sqrt{\nu})\sqrt{C\tau})^2}\left(\|\varphi\|_{L_2(0,l)} + 2\|f\|_{2,1,Q_\tau}\right), \quad \tau < \tau_1. \tag{3.11}$$

Split the segment $[0,T]$ into the intervals $\triangle_1 = \left[0, \frac{\tau_1}{2}\right]$, $\triangle_2 = \left[\frac{\tau_1}{2}, \tau_1\right]$, …, $\triangle_N$ with the length less than $\frac{\tau_1}{2}$. Then the estimate (3.11) is valid for each interval $\triangle_i$ and

$$|u|_{Q_\tau} \le \frac{(1+\sqrt{\nu})^2}{(\sqrt{\nu} - (1+\sqrt{\nu})\sqrt{C\tau})^2}\left(\|\varphi\|_{L_2(\Omega)} + 2\|f\|_{2,1,Q_\tau}\right), \quad \forall\tau \in [0,T].$$

Finally, we have

$$|u|_{Q_\tau} \le F(\tau), \quad \forall\tau \in [0,T], \tag{3.12}$$

where $F(\tau) = \dfrac{(1+\sqrt{\nu})^2}{(\sqrt{\nu} - (1+\sqrt{\nu})\sqrt{C\tau})^2}\left(\|\varphi\|_{L_2(\Omega)} + 2\|f\|_{2,1,Q_\tau}\right)$.

The inequality (3.12) implies that there exists at most one solution of the problem (2.1)–(2.4).

2. Existence. Let $\{\varphi_k(x)\} \in C^2(\Omega)$ be a basis in $W_2^2(\Omega)$, We define approximations $u^N(x,t)$ by

$$u^N(x,t) = \sum_{k=1}^N c_k^N(t)\varphi_k(x), \tag{3.13}$$

where $c_k(t)$ are solutions to the Cauchy problem

$$\frac{d}{dt}(u^N, \varphi_m) + (a(x,t)u_x^N, \varphi_{m\,x}) + (u^N, \varphi_m) -$$

$$-\left(\frac{1}{l}\int_0^l u^N(s,t)\,ds, \varphi_m\right) = (f, \varphi_m), m = 1, \ldots, N, \tag{3.14}$$

$$c_m^N(0) = (\varphi, \varphi_m). \tag{3.15}$$

We write the Cauchy problem (3.14)-(3.15) such that

$$\frac{d}{dt}c_m^N(t) + \sum_{k=1}^{N} c_k^N G_{km}(t) = f_m(t), \tag{3.16}$$

where

$$G_{km}(t) = \int_0^l (a\varphi_{k\,x}(x)\varphi_{m\,x}(x) + \varphi_k(x)\varphi_m(x))\,dx - \frac{1}{l}\left(\int_0^l \varphi_k(x)\,dx\right)\left(\int_0^l \varphi_m(x)\,dx\right),$$

$$f_m(t) = \int_0^l f(x,t)\varphi_m(x)\,dx.$$

Under the hypothesis of the theorem coefficients G_{km} are bounded and $f_k \in L_1(0,T)$. Thus the Cauchy problem has a unique solution $c_k \in W_2^1(0,T)$ for every$N$ and all the approximations (3.13) are defined.

Multiplying (3.16) by $\varphi_m(x)$, summing up from $k = 1$ to $k = N$ and integrating with respect to t from 0 to τ, we obtain the equality (3.1) for the functions $\{u^N\}$.As it was shown in the first part this implies that

$$|u^N|_{V_2(Q_T)} \le F(T),$$

where $F(T) > 0$ and does not depend on N.

Next, it is nessesary to show that we can extract from the sequence u^N a subsequence which converges weakly in $L_2(Q_T)$ and uniformly with respect to $t \in [0,T]$. To this end prove that the functions $l_{N,k}(t) = (v^N(x,t), \varphi_k(x))$ converge uniformly on $[0,T]$.

The estimate (3.12) implies that the functions $l_{N,k}(t)$ are bounded uniformly. We need to prove that $l_{N,k}(t)$ are equicontinuous on $[0,T]$ with a fixed k and an arbitrary $N \ge k$, that is, $|l_{N,k}(t+\triangle t) - l_{N,k}(t)| \to 0$ as $\triangle t \to 0$.

From (3.14) we have

$$|l_{N,k}(t+\triangle t) - l_{N,k}(t)| \le \int_t^{t+\triangle t} \left|(au_x^N, \varphi_{k\,x})\right| dt + \int_t^{t+\triangle t} \left(\left|(u^N, \varphi_k)\right| + |(f, \varphi_k)|\right) dt +$$

$$+\left|\frac{1}{l}\int_0^l \int_0^{t+\triangle t}\int_0^l u^N(s,t)\,ds\,dt\varphi_k(x)\,dx\right| \le \varepsilon(\triangle t)\|\varphi_k\|,$$

where $\varepsilon(\triangle t) \to 0$ as $\triangle t \to 0$ and does not depend on N. Hence, the functions $l_{N,k},\ N = k,\ k{+}1,\ \ldots,$ are equicontinuous with respect to t and furthermore there exists a subsequence $l_{N_r,k}$ that converges uniformly on $[0,T]$ to some continuous function $l_k(t)$ for each k.

We define $u = \sum_{k=1}^{\infty} l_k(t)\varphi_k(x)$ and show that a subsequence u^{N_r} converges to $u(x,t)$ weakly in $L_2(\Omega)$ and uniformly with respect to $t \in [0,T]$.

Let $\varphi(x) \in L_2(\Omega)$. Consider

$$(u^{N_r} - u, \varphi) = \left(u^{N_r} - v, \sum_{k=1}^{\infty}(\varphi, \varphi_k)\varphi_k\right) = \sum_{k=1}^{s}(\varphi, \varphi_k)(u^{N_r} - u, \varphi_k) +$$

$$+\left(u^{N_r}-u,\ \sum_{k=s+1}^{\infty}(\varphi,\varphi_k)\varphi_k\right). \tag{3.17}$$

Applying the Cauchy inequality we obtain

$$\left|\left(u^{N_r}-u,\ \sum_{k=s+1}^{\infty}(\varphi,\varphi_k)\varphi_k\right)\right|^2 \le \|u^{N_r}-u\|^2\left\|\sum_{k=s+1}^{\infty}(\varphi,\varphi_k)\varphi_k\right\|^2 =$$

$$=\|u^{N_r}-u\|^2\left(\sum_{k=s+1}^{\infty}(\varphi,\varphi_k)\varphi_k,\ \sum_{k=s+1}^{\infty}(\varphi,\varphi_k)\varphi_k\right)=$$

$$=\|u^{N_r}-u\|^2\left(\sum_{k=s+1}^{\infty}(\varphi,\varphi_k)^2\right)\le MR(s),$$

where $R(s)$ is the residial of the convergent Fourier series and M does not depend on N_r.

Then for a given $\varepsilon>0$ there exists such s that $MR(s)<\varepsilon$.

Furthermore,

$$\sum_{k=1}^{s}(\varphi,\varphi_k)(u^{N_r}-u,\varphi_k)=\sum_{k=1}^{s}(\varphi,\varphi_k)((u^{N_r},\varphi_k)-(u,\varphi_k))=$$

$$=\sum_{k=1}^{s}(\varphi,\varphi_k)(l_{N_r,k}-l_k).$$

As $l_{N_r,k}$ converges uniformly to l_k on $[0,T]$, so for a fixed s the sum $\sum_{k=1}^{s}(\varphi,\varphi_k)(l_{N_l,k}-l_k)<\varepsilon$ for all $t\in[0,T]$.

Hence, $|(u^{N_r}-u,\varphi)|<\varepsilon$ for all $t\in[0,T]$ and the sequaence u^{N_r} converges weakly to $u(x,t)$ in $L_2(\Omega)$ and uniformly with respect to $t\in[0,T]$.

Therefore, there exists a subsequence of $\{u^{N_r}\}$ which weakly converges to u in $L_2(Q_T)$ together with its derivatives $u_x^{N_r}$. Using the weak convergence one concludes that the limit function also satisfies $|u|_{Q_T}\le Const$ and $u\in V_2(Q_T)$.

We need only to show that this limit function u is a required generalized solution. To show that (2.5) is valid we multiply (3.14) by a smooth function $d_m(t)$, $d_m(T)=0$, take the sum from $m=1$ to $m=N'<N$ and integrate with respect to t from 0 to T. This leads us to the equality

$$\int_{Q_T}\left(-u^N\Psi_t^{N'}+au_x^N\Psi_x^{N'}+u^N\Psi^{N'}-\Phi^N\Psi^{N'}\right)dx\,dt=\int_{Q_T}f\Psi^{N'}\,dx\,dt+\int_\Omega\varphi\Psi^{N'}(x,0)\,dx, \tag{3.18}$$

where $\Psi^{N'}(x,t)=\sum_{m=1}^{N'}d_m(t)\varphi_m(x)$.

Taking into account the proved above convergence one can pass to the limit in (3.18) as $N_r\to\infty$ for fixed N' and obtain the equality (2.5 for the function $u(x,t)\in V_2(Q_T)$.

$$\int_{Q_T}\left(-v\Psi_t^{N'}+a_{ij}v_{x_i}\Psi_{x_j}^{N'}+av\Psi^{N'}-\Phi\Psi_t^{N'}+a_{ij}\Phi_{x_i}\Psi_{x_j}^{N'}+a\Phi\Psi^{N'}\right)dx\,dt=$$

$$=\int_{Q_T}f\Psi^{N'}\,dx\,dt+\int_\Omega\varphi\Psi^{N'}(x,0)\,dx. \tag{3.19}$$

As the set of functions $\Phi=\bigcup_{N'=1}^{\infty}\Phi^{N'}$ is dense in $\widehat{W}_2^1(Q_T)$, it follows that the limit relation (3.19) is fulfilled for every function $\Phi(x,t)\in\widehat{W}_2^1(Q_T)$ and hence, u is the solution of the problem (2.1)–(2.4). □

4 Conclusions

This paper describes the problem which arises from the application of the MAC method in mechanics. The MAC model for the parabolic equation which has the form of the integro-differential equation is considered. A generalized solution of the problem is studied. The main results are the obtained uniqueness and existence of the generalized solution. To prove these results we use Galerkin method and method of apriory estimates.

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Competing Interests

The authors declare that no competing interests exist.

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