British Journal of Mathematics & Computer Science
10(2): 1-8, 2015, Article no.BJMCS.18825
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

Solving Interval Linear Equations with Modified Interval Arithmetic

Mehdi Allahdadi[1*] and Zohreh Khorram[1]

[1] *Mathematics Faculty, University of Sistan and Baluchestan, Zahedan, Iran.*

Article Information

DOI: 10.9734/BJMCS/2015/18825
Editor(s):
(1) Drago - Patru Covei, Department of Mathematics, University Constantin Brancusi of Targu-Jiu, Romania.
Reviewers:
(1) Anonymous, Rajshahi University of Engineering and Technology, Bangladesh.
(2) Anonymous, Jaypee Institute of Information Technology, Noida, India.
(3) Anonymous, Kwame Nkrumah University of Science and Technology, Ghana.
(4)Anonymous, University of Namk Kemal, Turkiye.
Complete Peer review History: http://sciencedomain.org/review-history/10078

Short Research Article

Received: 11 May 2015
Accepted: 05 June 2015
Published: 07 July 2015

Abstract

Gaussian elimination method is one of the widely used methods for solving linear equations. An interval version of Gaussian elimination method has been used by simply replacing each real arithmetic step by the corresponding interval arithmetic step. Two interval arithmetics technique has been considered for modified interval arithmetics as well as several existing interval arithmetics. In this paper, modified interval arithmetic has been introduced based on two interval arithmetics technique. If we solve interval linear system of equations by existing interval arithmetic method the replacing solution in interval system of equations, the interval width is more than the interval width of right hand side intervals. On the other hand, applying modified interval arithmetic the interval width is less than interval width than previously obtained by existing interval arithmetic. Moreover, the closeness of interval width in system of equations to the right hand side is important so modified interval arithmetic is more effective and efficient for solving interval linear system of equations.

Keywords: Existing interval arithmetic; interval linear equations; modified interval arithmetic.

2010 Mathematics Subject Classification: 65G40; 15A06

**Corresponding author: E-mail: m_allahdadi@math.usb.ac.ir*

1 Introduction

Computing solution of the usual linear system of equations is easy, but obtaining the solution of interval linear equations is much more complicated. In fact, Obtaining solution of interval linear equations is a challenging problem in interval analysis. This problem was first considered by [1]. Some researchers worked on uncertain linear system. They proposed several methods for solving linear equations under uncertainty. Some of them focused on fuzzy linear equations [2]. Some others found an enclosure of solutions of interval linear system ([3]-[6]).

There are several methods for solving linear equations. Gaussian elimination method is one of them. An interval version of Gaussian elimination method can be obtained from simply replacing each real arithmetic step by the corresponding interval arithmetic step.
Firstly, we review some definitions.

An interval number X is generally represented as $[\underline{X}, \overline{X}]$ where $\underline{X} \leq \overline{X}$. If $\underline{X} = \overline{X}$, then X will be degenerate.

If $\underline{A}$ and $\overline{A}$ are two matrices in $\mathbb{R}^{m\times n}$ and $\underline{A} \leq \overline{A}$, then the set of matrices

$$\mathbf{A}^I = [\underline{A}, \overline{A}] = \{A|\ \underline{A} \leq A \leq \overline{A}\},$$

is called an interval matrix, and the matrices $\underline{A}$ and $\overline{A}$ are called its bounds. Center and radius matrices have been defined as

$$A^c = \frac{1}{2}(\underline{A} + \overline{A}) \quad , \quad \Delta_{\mathbf{A}} = \frac{1}{2}(\overline{A} - \underline{A}).$$

A square interval matrix $\mathbf{A}^I$ is called regular if each $A \in \mathbf{A}$ is nonsingular.

A special case of an interval matrix is an interval vector which is a one-column interval matrix

$$\mathbf{x}^I = \{\mathbf{x}|\ \underline{\mathbf{x}} \leq \mathbf{x} \leq \overline{\mathbf{x}}\},$$

where $\underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathbb{R}^n$.

Interval arithmetic is defined in [7].

Let $\mathbf{A}^I$ and $\mathbf{b}^I$ be $n \times n$ interval matrix and interval vector,respectively. For an interval system of linear algebraic equations $\mathbf{A}^I\mathbf{x} = \mathbf{b}^I$, the solution set is defined as

$$S = \{\mathbf{x} \in \mathbb{R}^n |\ A\mathbf{x} = b, A \in \mathbf{A}^I, b \in \mathbf{b}^I\}.$$

If $\mathbf{A}^I$ is regular, then for a matrix $A \in \mathbf{A}^I$ and any vector $b \in \mathbf{b}^I$ an ordinary linear system $A\mathbf{x} = b$ has the unique solution.

The solution set S is a non-convex polyhedral set [1]. The main objective is to find interval solution of linear interval system that is to determine The narrowest interval vector containing the set S is called the interval hull of S. This set is generally not an interval vector. It is usually difficult to describe S. Because S is generally so complicated in shape [8], it is usually impractical to use it. Instead, it is a common practice to seek the interval vector $\mathbf{x}^I$ containing S that the narrowest possible interval components. This interval vector is called the hull of the solution set or simply the hull. In other words, we need to imbed the solution set S into the minimal box in $\mathbb{R}^n$. This problem is known to be NP-hard [9] and complicated from computational viewpoint for large-scale systems. Therefore, many researchers generally compute only outer bounds for the hull ([4],[5],[7],[10]-[15]). Some iterative approaches were established at this context as well as direct numerical methods that provide over bounding of $\mathbf{x}^*$ ([6],[15]).

Suppose we solve interval linear system of equations by Gaussian elimination method with existing interval arithmetic. If we replace the obtained solution in interval equations of system, then the width of obtained interval is more than the width of the right hand side interval. Therefore, if closeness of obtained interval to the right hand side interval is important, then we can use modified interval arithmetic for solving interval linear system of equations. In this paper, we solve interval linear system of equations by Gaussian elimination with modified interval arithmetic and we show that resulted interval of substituting solution obtained through modified interval arithmetic cannot give rise to excess width than the existing interval arithmetic.

2 Review of Interval Arithmetic

In this section, we recall the existing and modified interval arithmetic.

2.1 Existing Interval Arithmetic

In this section, we review the existing interval arithmetic.
If $\bullet \in \{+, -, \times, \div\}$ denotes any one of these operations for arithmetic on real numbers x and y, then the corresponding operation for existing arithmetic on interval numbers X and Y is

$$X \bullet Y = \{x \bullet y | x \in X, y \in Y\}.$$

Therefore, this definition produces the following rules for generating the endpoint of $X \bullet Y$.

$$\begin{aligned}
&X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}],\\
&X - Y = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}],\\
&X \times Y = [\min(\underline{X}\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y}), \max(\underline{X}\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y})],\\
&X \div Y = X \times (\frac{1}{Y}) = X \times [\frac{1}{\overline{Y}}, \frac{1}{\underline{Y}}],
\end{aligned}$$

we exclude division by an interval containing 0.

2.2 Modified Interval Arithmetic

In this section, we recall the modified interval arithmetic [16].
Let $* \in \{\oplus, \ominus, \otimes, \oslash\}$ be any one of the operations for arithmetic on real numbers x and y, then the corresponding operation for modified arithmetic on interval numbers X and Y is

$$X * Y = [m(X) * m(Y) - k, m(X) * m(Y) + k],$$

where $k = \min\{(m(X) * m(Y)) - \alpha, \beta - (m(X) * m(Y))\}$ and $m(\cdot)$ is the midpoint of interval. α and β are the endpoint of the interval $X \bullet Y$.

$$
\begin{aligned}
X \oplus Y &= [m(X) + m(Y) - k, m(X) + m(Y) + k],\\
&\quad k = \frac{(\overline{X} + \overline{Y}) - (\underline{X} + \underline{Y})}{2},\\
X \ominus Y &= [m(X) - m(Y) - k, m(X) - m(Y) + k],\\
&\quad k = \frac{(\overline{X} + \overline{Y}) - (\underline{X} + \underline{Y})}{2},\\
X \otimes Y &= [m(X)m(Y) - k, m(X)m(Y) + k],\\
&\quad k = \min\{(m(X)m(Y)) - \alpha, \beta - (m(X)m(Y))\},\\
&\qquad \alpha = \min(\underline{X}\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y}),\\
&\qquad \beta = \max(\underline{X}\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y}).\\
X^{-1} &= [\frac{1}{m(X)} - k, \frac{1}{m(X)} + k],\\
&\quad k = \min\{\frac{1}{\overline{X}}(\frac{\overline{X} - \underline{X}}{\overline{X} + \underline{X}}), \frac{1}{\underline{X}}(\frac{\overline{X} - \underline{X}}{\overline{X} + \underline{X}})\},\ 0 \notin X.
\end{aligned}
$$

It is to be noted that $X * Y \subseteq X \bullet Y$ [16]. For example if we take $X = [-1, 2]$ and $Y = [3, 5]$, then $X \times Y = [-5, 10]$ whereas $X \otimes Y = [-5, 9]$.

3 Gaussian Elimination Method

There are several methods for solving linear equations that can be labeled as Gaussian elimination. An interval version of any of them can be obtained from one using ordinary real arithmetic by simple replacing each real arithmetic step by the corresponding interval arithmetic step. One standard method involves converting the coefficient matrix to the upper triangular matrix and hence the interval solution vector computed using Gaussian elimination contains the set S. Suppose the elimination procedure does not fail because of division by an interval containing zero. The solution X^I computed by using interval Gaussian elimination contain S [8].

Unfortunately, Using an interval version of classic Gaussian elimination generally does not yield a suitable algorithm because of

- rounding errors in interval computations.
- division by an interval containing zero.
- dependence among generated intervals.

Now, Let X^{EI} and X^{MI} be solution obtained through solving interval linear equations $\mathbf{A}^I\mathbf{x} = \mathbf{b}^I$ with the existing and modified interval arithmetic, respectively. Also, let $\mathbf{b}^{EI}$ and $\mathbf{b}^{MI}$ be intervals resulted from replacing X^{EI} and X^{MI} in equations of system, respectively. We define

$$\varepsilon^{EI} = \mathbf{b}^I - \mathbf{b}^{EI}\ ,\ \varepsilon^{MI} = \mathbf{b}^I - \mathbf{b}^{MI}.$$

For example, consider the equation $[1, 2]X = [4, 5]$.

$$X^{EI} = [4, 5] \div [1, 2] = [2, 5] \quad , \quad X^{MI} = [4, 5] \oslash [1, 2] = [2, 4],$$

therefore

$$
\begin{aligned}
&[1, 2]X^{EI} = [2, 10] = \mathbf{b}^{EI}\ ,\ [1, 2]X^{MI} = [2, 7] = \mathbf{b}^{MI},\\
&\varepsilon^{EI} = [4, 5] - [2, 10] = [-6, 3] \quad , \quad \varepsilon^{MI} = [4, 5] \ominus [2, 7] = [-3, 3],
\end{aligned}
$$

and hence

$$[1, 2]X^{EI} \supseteq [1, 2]X^{MI} \supseteq [4, 5].$$

4 Numerical Examples

To demonstrate the effectiveness of the proposed approach in solving the interval systems of linear equations, the section presents a comparative study on the relative performance of the proposed method with the other approach. Some interval computations have been done by the interval toolbox INTLAB v6 [17].

Example 4.1. Consider an interval linear system of equations

$$\begin{cases} [2,3]x_1 + [0,1]x_2 = [0,120] \\ [1,2]x_1 + [2,3]x_2 = [60,240] \end{cases}$$

Solutions of this system with existing and modified interval arithmetic are given in Table 1 on page 6. From Table 1, we conclude the solution obtained through modified interval arithmetic is closer to the right hand side intervals of system than existing interval arithmetic and hence the norm of error will be smaller than that one. Exact solution and the solution resulted from the existing and modified interval arithmetic is shown in Fig. 1 on page 104.

Example 4.2. Consider an interval system of linear algebraic equations

$$\begin{cases} [2,3]x_1 + [1.5,2]x_2 + [3,3.4]x_3 = [15,20] \\ [4.5,5.5]x_1 + [2.5,3.5]x_2 + [-1.5,-1]x_3 = [5,10] \\ [1,1.6]x_1 + [-7,-6]x_2 + [2,3]x_3 = [4,6]. \end{cases}$$

Solutions of this system with existing and modified interval arithmetic are given in Table 2 on page 6.

By considering Table 2, the solution resulted from modified interval arithmetic is closer to the right hand side intervals of system than the existing interval arithmetic and hence the norm of error will be smaller.

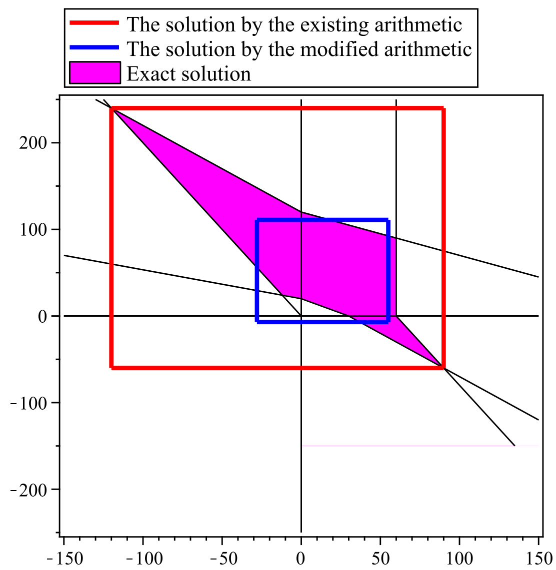

Figure 1: Solution of example 4.1

Table 1: Solutions and errors of example 4.1

X^{EI}	$\begin{pmatrix} [-120, 90] \\ [-60, 240] \end{pmatrix}$
X^{MI}	$\begin{pmatrix} [-27.406, 54.679] \\ [-6.909, 110.546] \end{pmatrix}$
b^{EI}	$\begin{pmatrix} [-420, 510] \\ [-420, 900] \end{pmatrix}$
b^{MI}	$\begin{pmatrix} [-89.127, 209.127] \\ [-75.539, 375.539] \end{pmatrix}$
$\|\varepsilon^{EI}\|$	$[0, 998.599]$
$\|\varepsilon^{MI}\|$	$[0, 378.549]$

Table 2: Solutions and errors of example 4.2

X^{EI}	$\begin{pmatrix} [-1.5611, 4.9758] \\ [-0.2695, 3.5165] \\ [1.5360, 6.5495] \end{pmatrix}$
X^{MI}	$\begin{pmatrix} [0.8993, 2.7248] \\ [-0.1596, 2.0633] \\ [1.8500, 5.2151] \end{pmatrix}$
b^{EI}	$\begin{pmatrix} [-4.1358, 44.2283] \\ [-22.6841, 39.3915] \\ [-27.5717, 31.1196] \end{pmatrix}$
b^{MI}	$\begin{pmatrix} [7.1094, 27.8906] \\ [0.8728, 14.1272] \\ [2.0533, 7.9467] \end{pmatrix}$
$\|\varepsilon^{EI}\|$	$[0, 56.2506]$
$\|\varepsilon^{MI}\|$	$[0, 16.2803]$

5 Conclusion

There are several methods for solving linear equations. Gaussian elimination method is one of them. An interval version of Gaussian elimination method can be obtained from simply replacing each real arithmetic step by the corresponding interval arithmetic step. In this study, interval Gaussian elimination method based on modified interval arithmetic has been considered. If we solve interval linear system of equations by the modified interval arithmetic, the replacing solution in interval system of equations, the interval width is less than the interval width previously obtained by existing interval arithmetic. Also, the closeness of interval width in system of equations to the right hand side is important so modified interval arithmetic is more effective for solving interval linear system of equations.

Acknowledgement

We would like to thank anonymous referees for their constructive comments and suggestions that helped to improve the paper.

Competing Interests

The authors declare that no competing interests exist.

References

[1] Oettli W, Prager W. Compatibility of approximate solution of linear equations with given error bounds for coefficients and right hand sides. Numer. Math. 1964;6:405-409.

[2] Nayak S, Chakraverty S. A new Approch to solve Fuzzy system of linear Equation, contents list available at tjmcs. Journal of Mathematics and Computer Science. 2013;7:205-212.

[3] Abolmasoumi S, Alavi M. A method for calculating interval linear system. Journal of Mathematics and Computer Science. 2014;8:193-204.

[4] Neumaier A. The enclosure of solutions of parameter-dependent systems of equations, in Moore; 1988.

[5] Polyak BT, Nazin SA. Interval solutions for interval algebraic equations. Mathematics and Computers in Simulation. 2004;66:207-217.

[6] Rohn J. Overstimations in bounding solutions of perturbed linear equations. Linear Algebra Appl. 1997;262:55-65.

[7] Alefeld G, Herzberger J. Introduction to interval computations. Academic Press, Orlando, Florida; 1983.

[8] Hansen E, William G, Walster G. Global optimization using interval analysis. Second Edition, Revised and Expanded, New York; 2004.

[9] Kreinovich V, Lakeev AV, Noskov SI. Optimal solution of interval linear systems is intractable (NP-hard). Interval Comput. 1993;1:6-14.

[10] Hansen ER. The hull of preconditioned interval linear equations. Reliable Computing. 2000;6:95-103.

[11] Hladik M. Solution set characterization of linear interval systems with a specific dependence structure. Reliable Computing. 2007;13:361-374.

[12] Kearfott R. Rigorous global search: Continuous problems. Kluwer Academic Publ., Dordrecht; 1996.

[13] Neumaier A. Interval methods for system of equations. Cambridge University Press, London; 1990.

[14] Neumaier A. Linear interval equations, in Nickel; 1986.

[15] Shary SP. On optimal solution of interval linear equations. SIAM J. Numer. Anal. 1995;32:610-630.

[16] Ganesan K, Veeramani P. On arithmetic operations of interval numbers. International Journal of Unccertainty, Fuzziness and Knowledge - Based Systems. 2005;13:619-631.

[17] Rump SM. Intlab-interval laboratory. In: Csendes, T. (ed.) Developments in reliable computing. Kluwer Academic Publishers, Dordrecht. 1999;77-104.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/10078