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Some Commutativity Theorems for Prime Near-rings Involving Derivations

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Authors' contributions

This work was carried out in collaboration between two authors. Both authors read and approved the final manuscript.

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Abstract

The study depicts that a prime near-ring N is considered to be a commutative ring if there non-negative integers exist i.e., $p \geq 0, q \geq 0$ in such a way that N admits a non-zero derivation, where d satisfying one of the conditions like $(c_1) - (c_8)$. For any $x, y \in N,$ we define the following properties

(c_1) $d([x,y]) - x^p(xoy)x^q = 0$;

(c_2) $d([x,y]) + x^p(xoy)x^q = 0$;

(c_3) $d(xoy) - x^p([x,y])x^q = 0$;

(c_4) $d(xoy) + x^p([x,y])x^q = 0$;

(c_5) $d([x,y]) - y^p(xoy)y^q = 0$;

(c_6) $d([x,y]) + y^p(xoy)y^q = 0$;

(c_7) $d(xoy) - y^p([x,y])y^q = 0$;

(c_8) $d(xoy) + y^p([x,y])y^q = 0$.

In addition, an example is given to demonstrate the primeness of the hypothesis which is not superfluous. Finally, we can conclude it with some open problems.

Keywords: Commutativity; derivation; near-ring; prime near-ring; zero-symmetric near-ring.

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1 Introduction

In all that follows a right near-ring N is a non-empty set with two operations $+$ and $\cdot$ such that $(N,+)$ is a group and $(N,\cdot)$ is a semi group satisfying the right distributive law $(y+z)\cdot x = y\cdot x + z\cdot x$ for all $x,y,z \in N$. A right near-ring N is zero symmetric if $x\cdot 0 = 0$ for all $x\in N$, (see Pilz [1] for details), recall that right distributivity yields $0\cdot x = 0$). Throughout the paper, we will use the word near-ring to mean zero symmetric right near-ring and denote xy instead of $x\cdot y$. According to Bell and Mason [2], a near-ring N is said to be prime if $xNy = \{0\}$ for $x,y\in N$ implies $x=0$ or $y=0$. An additive mapping $d: N\to N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ (or equivalently, as noted in Wang [3], that $d(xy) = d(x)y + xd(y)$ for all $x,y\in N$. The symbol $Z(N)$ will represent the multiplicative center of N, that is, $Z(N) = \{x\in N | xy = yx \text{ for all } y\in N\}$. Note that $Z(N)$ is a non-empty set, that is $Z(N)\neq \emptyset$, since $0\in Z(N)$. For any $x,y\in N$, the symbol $[x,y]$ stands for the commutator $xy-yx$, while the symbol xoy will denote the anti commutator $x\,y+yx$. There has been a great deal of work concerning commutativity of prime and semi prime rings with derivations satisfying certain differential identities stating that the existence of a suitably constrained on a prime near-ring forces the near-ring to be a commutative ring (see [4-10] for references). Many results asserting that prime near-ring with certain constrained derivations have ring like behavior. Several results in literature demonstrate that "how the structure of a ring is connected with the additive mapping defined on that ring." Many authors [11,12,13] have studied the structure of prime and semi prime rings admitting suitably constrained additive mappings, as automorphisms, derivations, skew-derivations and generalized derivations acting on appropriate subsets of the ring. Motivated by these observations, it is a natural to look for comparable results as near-ring. Our aim in this paper is to extend some results on prime near-ring with non-zero derivation satisfying some differential identities to become a commutative ring, and it is organized as follows. In Section 2, we present our main theorems, Section 3 devotes a counterexample, Section 4 includes conclusion and finally, Section 5 provides some open problems.

2 Main Results

The main results of this paper are as given below.

Theorem 2.1. Let N be a prime near-ring and there exist nonnegative integers $p\geq 0, q\geq 0$. If N admits a non-zero derivation d such that either (c_1) $d(xy-yx) - x^p(xy+yx)x^q = 0$ or (c_2) $d(xy-yx) + x^p(xy+yx)x^q = 0$ for any $x,y\in N$, then N is a commutative ring.

Theorem 2.2. Let N be a prime near-ring and there exist nonnegative integers $p\geq 0, q\geq 0$. If N admits a non-zero derivation d such that either (c_3) $d(xy+yx) - x^p(xy-yx)x^q = 0$ or (c_4) $d(xy+yx) + x^p(xy-yx)x^q = 0$ for any $x,y\in N$, then N is a commutative ring.

Theorem 2.3. Let N be a prime near-ring and there exist nonnegative integers $p\geq 0, q\geq 0$. If N admits a non-zero derivation d such that (c_5) $d(xy-yx) - y^p(xy+yx)y^q = 0$ or (c_6) $d(xy-yx) + y^p(xy+yx)y^q = 0$ for any $x,y\in N$, then N is a commutative ring.

Theorem 2.4. Let N be a prime near-ring and there exist nonnegative integers $p\geq 0, q\geq 0$. If N admits a non-zero derivation d such that (c_7) $d(xy+yx) - y^p(xy-yx)y^q = 0$ or (c_8) $d(xy+yx) + y^p(xy-yx)y^q = 0$ for any $x,y\in N$, then N is a commutative ring.

In order to prove our main results, we begin with the following known and elementary Facts.

Fact 2.5 [14]**.** Taking a prime near-ring N that admits a non-zero derivation d with $d(N)\subset Z(N)$. then N is a commutative ring.

Fact 2.6. Let N be and a prime near-ring. Then, for any $x, y \in N$:

(i) [x, y x] = [x, y] x ; (ii) x o (yx) = (xo y) x
(iii) [xy, y] = [x, y] y ; (iv) (xy) o y = (xo y) y.

Proof of Theorem 2.1. By the hypotheses (c_1), we have

$$d([x,y]) = x^p(xy+yx)x^q \quad \forall\, x,y \in N. \tag{2.1}$$

Taking y by yx in (2.1) and using the Fact 2.6 (i) and (ii), we find that

$$d([x,y]x) = x^p(xy+yx))x^{q+1} \quad \forall\, x,y \in N. \tag{2.2}$$

In view of a non-zero derivation d, one can write

$$d([x,y]x) = d([x,y])x + [x,y]\,d(x). \tag{2.3}$$

Combining Equations (2.1) and (2.2) in (2.3), we obtain

$$x^p(xy+yx)x^{q+1} = x^p(xy+yx)x^{q+1} + [x,y]d(x).$$

This implies that

$$[x,y]d(x) = 0.$$

But rest of the proof follows immediately from Theorem 2.2 (i) in [15].

Next, from(c_2), we have

$$d([x,y]) = -x^p(xy+yx)x^q, \ \ x,y \in N. \tag{2.4}$$

Replacing y by yx and using Fact 2.6 (i) and (ii), we have

$$d([x,y]x) = -x^p(xy+yx)x^{q+1}. \tag{2.5}$$

By definition of non-zero derivation d, we have

$$d([x,y]x) = d([x,y])x + [x,y]\,d(x) \tag{2.6}$$

Use the obtained results of (2.4) and (2.5) in (2.6) to get

$$-x^p(xy-yx)x^{q+1} = \ -x^p(xy-yx)x^{q+1} + [x,y]d(x).$$

This gives

$$[x,y]d(x) = 0 \quad \text{for all } x,y \in N.$$

Next, the remaining proof of this result is same as the proof of Theorem 2.2(ii) in [15].

The following results are the corollaries of our Theorem 2.1.

Corollary 2.1.1 ([15, Theorem 2.2]). Let N be a prime near-ring and there exist nonnegative integers $p \geq 0, q \geq 0$. If N admits a non-zero derivation d such that either $d(xy - yx) - x^p(xy - yx)x^q = 0$ or $d(xy - yx) + x^p(xy - yx)x^q = 0$ for any $x, y \in N$, then N is a commutative ring.

Corollary 2.1.2. Let N be a prime near-ring and there exist nonnegative integers $t \geq 0$. If N admits a non-zero derivation d such that either $d(xy - yx) \pm (xy + yx)x^t = 0$ or $d(xy - yx) \pm x^t(xy + yx) = 0$ for any $x, y \in N$, then N is a commutative ring.

Proof of Theorem 2.2. By hypotheses (c_3), we have

$$d(xoy) = x^p(xy - yx)x^q \text{ , } \forall\, x, y \in N \tag{2.7}$$

Substituting y by yx in (2.7) and using the Fact 2.6 (i) and (ii), we find that

$$d(xoy)x = x^p(xy - yx)x^{q+1} \quad \forall\, x, y \in N. \tag{2.8}$$

In view of a derivation d, one can write

$$d\big((xoy)x\big) = d(xoy)x + (xoy)\, d(x) \tag{2.9}$$

Putting the results of (2.7) and (2.8) in (2.9), we get

$$x^p(xy - yx)x^{q+1} = x^p(xy - yx)x^{q+1} + (xoy)d(x).$$

This implies that

$$\text{(x o y) } d(x) = 0 \quad \forall\, x, y \in N \qquad .$$

The rest of the proof follows immediately from proof of Theorem 2.4 (iii) in [15].

By hypotheses (c_4), we have

$$d(xoy) = -x^p(xy - yx)x^q \text{ , } \forall\, x, y \in N \text{ .} \tag{2.10}$$

Substituting y by yx in (2.10) and using the Fact 2.6 (i) and (ii), we find that

$$d(xoy)x = -x^p(xy - yx)x^{q+1} \quad \forall\, x, y \in N. \tag{2.11}$$

By definition of derivation d, we have

$$d\big((xoy)x\big) = d(xoy)x + (xoy)\, d(x) \text{ .} \tag{2.12}$$

Combining Equations (2.10) and (2.11) in (2.12), we get

$$-x^p(xy - yx)x^{q+1} = -x^p(xy - yx)x^{q+1} + (xoy)d(x).$$

This implies that

$$\text{(x o y) } d(x) = 0 \quad \forall\, x, y \in N.$$

The rest of the proof follows immediately from proof of Theorem 2.4 (iv) in [15].

As a consequence of Theorem 2.2, we get the main result of [15].

Corollary 2.2.1 ([15, Theorem 2.4]). Let N be a prime near-ring and there exist nonnegative integers $p \geq 0, q \geq 0$. If N admits a non-zero derivation d such that either $d(xy + yx) - x^p(xy + yx)x^q = 0$ or $d(xy + yx) + x^p(xy + yx)x^q = 0$ for any $x, y \in N$, then N is a commutative ring.

Corollary 2.2.2. Let N be a prime near-ring and there exist nonnegative integers $t \geq 0$. If N admits a non-zero derivation d such that either $d(xy + yx) \pm (xy - yx)x^t = 0$ or $d(xy + yx) \pm x^t(xy - yx) = 0$ for any $x, y \in N$, then N is a commutative ring.

Proof of Theorem 2.3. By hypotheses (c_5), we have

$$d([x, y]) = y^p(xy + yx)y^q \quad \forall\, x, y \in N. \tag{2.13}$$

Put x by xy in (2.13) and using the Fact 2.6 (iii) and (iv), we find that

$$d([x, y]y) = y^p(xy + yx))y^{q+1} \quad \forall\, x, y \in N. \tag{2.14}$$

By definition of d, one can write

$$d([x, y]y) = d([x, y])y + [x, y]\, d(y). \tag{2.15}$$

Combining Equations (2.13) and (2.14) in (2.15), we get

$$y^p(xy + yx)y^{q+1} = y^p(xy + yx)y^{q+1} + [x, y]d(y).$$

This implies that $\quad [x, y]d(y) = 0 \quad$ for all $\quad x, y \in N$

$$xyd(y) = yx\, d(y) \qquad \text{for all} \quad x, y \in N. \tag{2.16}$$

Putting x by wx in (2.16) and using (2.16), we find that

$$[x, y]\, w\, d(y) = 0 \ \ \forall\, x, y, w \in N. \tag{2.17}$$

This implies that

$$[x, y]\, N\, d(y) = 0 \ \ \forall\, x, y \in N. \tag{2.18}$$

Since N is a prime near-ring, so Equation (2.18) gives

$$\text{for each } \ y \in N,\ [x, y] = 0 \text{ or } d(y) = 0\, . \tag{2.19}$$

Clearly, if $[x, y] = 0$, $y \in Z(N)$. Consquently, if $y \in Z(N)$ then $d(y) \in Z(N)$. Thus, Equation (2.19) yields that for all $y \in N, d(y) \in Z(N)$, implies $d(N) \subset Z(N)$. In view of Fact 2.5, it gives that N is a commutative ring.

Now by the property (c_6), we have

$$d([x, y]) = -y^p(xy + yx)y^q \ \text{ for any } x, y \in N. \tag{2.20}$$

Setting x by xy in (2.20) and using the Fact 2.6 (iii) and (iv), we find that

$$d([x, y]y) = -y^p(xy + yx)y^{q+1} \text{ for all } x, y \in N. \tag{2.21}$$

By definition of derivation, we have

$$d([x,y]y) = d([x,y])y + [x,y]d(y). \tag{2.22}$$

Using Equation (2.20) and (2.21) in (2.22), we get

$$-y^p(xy - yx)y^{q+1} = -y^p(xy - yx)y^{q+1} + [x,y]d(y). \tag{2.23}$$

This implies that

$$[x,y]d(y) = 0 \quad \text{for all } x, y \in N. \tag{2.24}$$

The remaining proof is the same as above condition (c_5).

The following results are the immediate corollaries of Theorem 2.3.

Corollary 2.3.1.[15, Theorem 2.2] Let N be a prime near-ring and there exist nonnegative integers $p \geq 0, q \geq 0$. If N admits a non-zero derivation d such that either $d(xy - yx) - y^p(xy - yx)y^q = 0$ or $d(xy - yx) + y^p(xy - yx)y^q = 0$ for any $x, y \in N$, then N is a commutative ring.

Corollary 2.3.2. Let N be a prime near-ring and there exist nonnegative integers $t \geq 0$. If N admits a non-zero derivation d such that either $d(xy - yx) \pm (xy - yx)y^t = 0$ or $d(xy - yx) \pm y^t(xy - yx) = 0$ for any $x, y \in N$, then N is a commutative ring.

Proof of Theorem 2.4. By hypotheses (c_7), we have

$$d(xoy) = y^p(xy - yx)y^q \quad \forall\, x, y \in N. \tag{2.25}$$

Substituting x by xy in (2.25) and using the Fact 2.6 (i) and (ii), we find that

$$d(xoy)y = y^p(xy - yx)y^{q+1} \quad \forall\, x, y \in N. \tag{2.26}$$

By definition of derivation d, we have

$$d\big((xoy)y\big) = d(xoy)y + (xoy)\, d(y). \tag{2.27}$$

Substituting (2.25) and (2.26) in (2.27), we get

$$y^p(xy - yx)y^{q+1} = y^p(xy - yx)y^{q+1} + (xoy)d(y).$$

This implies that

$$(xoy)\, d(y) = 0 \ \ \forall\, x, y \in N$$

$$yxd(y) = -xy\, d(y) \qquad \forall \quad x, y \in N. \tag{2.28}$$

Replace x by t x in equation (2.28) and use (2.28) to obtain

$$ytxd(y) = -txyd(y) = (-t)\big(-yxd(y)\big) = (-t)(-y)xd(y) \ \ \forall\, x, y, t \in N.$$

or $\quad \big(yt - (-t)(-y)\big)xd(y) = 0\ \forall\ x, y, t \in N.$

Taking y by $-y$, then $(-yt + ty)xd(-y) = 0\ \forall\, x, y, t \in N.$

$$-[t,y]xd(-y) = 0 \;\forall\; x,y,t \in N.$$

This implies that

$$[t,y]xd(-y) = 0 \;\forall\; x,y,t \in N. \tag{2.29}$$

Since N is prime near–ring, we have for each $y \in N$,

$$d(y) = 0 \text{ or } \quad y \in Z(N). \tag{2.30}$$

We know that if $y \in Z(N)$, then $d(y) \in Z(N)$. Hence (2.30) forces that for all

$$y \in N, d(y) \in Z(N), \text{ that is, } d(N) \subset Z(N).$$

In view of the Fact 2.5, N is a commutative ring.

Next, we assume that the condition (c_8)

$$d(xoy) = -y^p(xy - yx)y^q \text{ for any } x,y \in N. \tag{2.31}$$

Putting x by xy in (2.31) and using the Fact 2.6 (iii) and (iv), we obtain

$$d\big((xoy)y\big) = -y^p(xy - yx)y^{q+1} \text{ for all } x,y \in N. \tag{2.32}$$

By definition of derivation d, we have

$$d\big((xoy)y\big) = d(xoy)y + (xoy)d(y). \tag{2.33}$$

Use (2.31) and (2.32) in (2.33) to get

$$-y^p(xy - yx)y^{q+1} = -y^p(xy - yx)y^{q+1} + (xoy)d(y). \tag{2.34}$$

This implies that $(xoy)d(y) = 0$ for all $x,y \in N$.

$$yx\, d(y) = -xy\, d(y) \quad \forall\, x,y \in N. \tag{2.35}$$

But (2.35) is the same as (2.28), arguing as in the above proof of Theorem 2.4 we reached that N is a commutative ring.

The following results are immediate corollaries of Theorem 2.4.

Corollary 2.4.1. Let N be a prime near-ring and there exist nonnegative integers $p \geq 0, q \geq 0$. If N admits a non-zero derivation d such that either $d(xy + yx) - y^p(xy + yx)y^q = 0$ or $d(xy + yx) + y^p(xy + yx)y^q = 0$ for any $x,y \in N$, then N is a commutative ring.

Corollary 2.4.2. Let N be a prime near-ring and there exist nonnegative integers $t \geq 0$. If N admits a non-zero derivation d such that either $d(xy + yx) \pm (xy - yx)y^t = 0$ or $d(xy + yx) \pm y^t(xy - yx) = 0$ for any $x,y \in N$, then N is a commutative ring.

3 Counterexample

The following example shows that the primeness hypothesis in Theorems 2.1, 2.2, 2.3 and 2.4 are necessary even in the case of arbitrary rings.

Example 3.1. Let R be a non-commutative ring and $N = \left\{ \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ 0 & 0 & \gamma \end{pmatrix} | \alpha, \beta, \gamma \in R \right\}$. Define a map $d: N \to N$ by $d\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, one can easily check that; d is a non-zero derivation on N. Let $B = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \alpha \neq 0$. Then $BNB = \{0\}$, which shows that N is not prime. In addition if d satisfies either $d([A,B]) = AoB$ or $d(AoB) = [A,B]$ for all $A, B \in N,$ and N is a non-commutative ring. Also, an alternate example can be found in [15].

4 Conclusion

In this paper we study some conditions to prove the commutativity of prime near–rings involving derivations. We conclude the paper by discussing some issues for future research work. The conditions (c_1) - (c_8) are assumed to be held for all x, y in N. Are Theorems 2.1-2.4 still true via generalized derivations or if these conditions hold for only x, y in S $\subseteq$ N, where S is a suitable non-zero ideal of N? Finally, we present some open problems.

5 Open Problems

One can look more general constraints on the derivation would be interesting.

5.1. Let N be a prime near-ring and there exist nonnegative integers $p \geq 0, q \geq 0$.
If N admits a non-zero generalized derivations d such that
$d(xy - yx) \pm x^p(xy + yx)x^q = 0$ or $d(xy - yx) \pm y^p(xy - yx)y^q = 0$,
for all $x, y \in N$, then N is a commutative ring.

5.2. Let N be a prime near-ring. If N admits a non-zero skew-derivation (skew-generalized) d such that either, any $x, y \in N$,
$d(xy - yx) \pm x^p(xy + yx)x^q = 0$ or $d(xy - yx) \pm y^p(xy + yx)y^q = 0$,
$p \geq 0, q \geq 0$ are integers, then N is a commutative ring.

5.3. Let N be a prime near-ring and there exist nonnegative integers $p \geq 0, q \geq 0$.
If N admits a non-zero multiplicative derivation (multiplicative generalized) d such that $d(xy - yx) \pm x^p(xy + yx)x^q \in Z(N)$ or
$d(xy - yx) \pm y^p(xy - yx)y^q \in Z(N)$ for any $x, y \in N$, then N is a commutative ring.

One can see the constraints such as commutativity of torsion free near-rings. The properties $(c_1) - (c_8)$ are assumed to be held for all $x, y \in N$. Do Theorems (2.1), 2.2, (2.3) or (2.4) true if these conditions held for only $x, y \in S \subset N$, where S is a non-zero ideal (semi group ideal) of prime near- ring?

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Competing Interests

Authors have declared that no competing interests exist.

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