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Nonlinear Sturm-Liouville Boundary Value Problems for Second Order Hamiltonian Systems with Impulsive Effects

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Author's contribution

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Abstract

We investigate nontrivial solutions for nonlinear Sturm-Liouville boundary value problems of second order Hamiltonian systems with impulsive effects and we obtain some new results.

Keywords: Sturm-Liouville boundary value problems; second order Hamiltonian system; impulsive effects; index theory.

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1 Introduction

We investigate the following problem

$$\ddot{x} + V'(t,x) = 0, t \in [0,1], t \neq t_j, j = 1,2,\cdots,p, \quad (1.1)$$
$$\Delta x'(t_j) = x'(t_j+0) - x'(t_j-0) = J_j(x'(t_j)), \quad (1.2)$$
$$x(0)\cos\alpha - x'(0)\sin\alpha = M_1(x(0),x'(0),x(1),x'(1)), \quad (1.3)$$
$$x(1)\cos\beta - x'(1)\sin\beta = M_2(x(0),x'(0),x(1),x'(1)), \quad (1.4)$$

where $V:[0,1]\times\mathbf{R}^n \to \mathbf{R}$, V'denotes the gradient of V with respect to x, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1, \Delta x'(t_j) = x'(t_j+0) - x'(t_j-0) = \lim_{t\to t_j^+} x'(t) - \lim_{t\to t_j^-} x'(t), J_j \in C(\mathbf{R}^n,\mathbf{R}^n)(j = 1,\cdots,p)$, $M_k \in C(\mathbf{R}^{4n},\mathbf{R}^n)(k=1,2)$ are bounded, $\alpha \in (0,\pi), \beta \in (0,\pi)$.

When $n = 1, \alpha = 0, \beta = \pi, J_j = 0, M_i = 0$ for $j = 1,\cdots,p$, $i = 1,2$, the problem (1.1)-(1.4) reduces to the following Duffing equation

$$\ddot{x} + f(t,x) = 0, \quad (1.5)$$
$$x(0) = 0 = x(1), \quad (1.6)$$

where $f(t,x) = V'(t,x)$. Under the condition

$$\frac{\partial}{\partial x} f(t,x) \leq -\eta < -\pi^2$$

for all $(t,x) \in I \times \mathbf{R}$ and some constant $\eta > 0$, [1] has proved that (1.5)-(1.6) has at least one solution. For the second Hamiltonian system

$$\ddot{x} + V'(t,x) = 0, \quad (1.7)$$
$$x(0)\cos\alpha - x'(0)\sin\alpha = 0, \quad (1.8)$$
$$x(1)\cos\beta - x'(1)\sin\beta = 0, \quad (1.9)$$

where $V \in C^1([0,1]\times\mathbf{R}^n,\mathbf{R}), \alpha \in [0,\pi)$ and $\beta \in (0,\pi]$, there exists the similar solvable condition (V_1) There exists $B \in L^\infty([0,1],\mathcal{L}_s(\mathbf{R}^n))$ such that

$$V'(t,x)\cdot x \leq B(t)x\cdot x, \forall (t,x) \in [0,1]\times\mathbf{R}^n \text{with} |x| \geq r.$$

In order to describe solvable conditions for (1.7)-(1.9) on B, we recall the index theory (see [2],[3]) for the linear systems (1.8)-(1.9) and

$$\ddot{x} + B(t)x = 0. \quad (1.10)$$

This index theory associates B with a pair of integers $(i_{\alpha,\beta}(B), \nu_{\alpha,\beta}(B)) \in \mathbf{N}\times\{0,1,\cdots,n\}$ as follows:

$$\nu_{\alpha,\beta}(B) = \text{ the dimension of the solution space of } (1.8)-(1.10),$$
$$i_{\alpha,\beta}(B) = \sum_{\lambda<0} \nu_{\alpha,\beta}(B+\lambda I_n). \quad (1.11)$$

In ([2],[3]) the author has proved that under condition (V_1) with $\nu_{\alpha,\beta}(B) = i_{\alpha,\beta}(B) = 0$ the problem (1.7)-(1.9) has one solution. Note that when $\alpha = 0, \beta = \pi$, (1.8)-(1.9) reduces to (1.6). As $\eta < \pi^2, i_{0,\pi}(\eta I_n) = 0 = \nu_{0,\pi}(\eta I_n)$. So that result generalizes Lees' result.

In this paper we further generalize the above results to the impulsive Hamiltonian system (1.1)-(1.4). To this end we also need the following assumptions:

(V_2) There exists $B_0 : [0,1] \times \mathbf{R}^n \to \mathcal{L}_s(\mathbf{R}^n)$ with $B_0(\cdot, x(\cdot)) \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ for $x(\cdot) \in Y$ (Y is defined in section 3) and $B_{01}, B_{02} \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ with $i_{\alpha,\beta}(B_{01}) = i_{\alpha,\beta}(B_{02}), \nu_{\alpha,\beta}(B_{02}) = 0$ such that

$$B_{01}(t) \leq B_0(t,x) \leq B_{02}(t), V'(t,x) = B_0(t,x)x \text{ as } |x| \to 0.$$

(M) $M_k(\xi) = o(|\xi|)$ as $|\xi| \to 0, k = 1, 2.$

$(J) J_j(\xi) = o(|\xi|)$ as $|\xi| \to 0, j = 1, 2, \cdots, p.$

The main result of this paper is the following theorem.

Theorem 1.1 Assume V satisfies (V_1) with $\nu_{\alpha,\beta}(B) = i_{\alpha,\beta}(B) = 0$. Then (1.1)-(1.4) has one solution. If further $(V_2), (M)$ and (J) hold, then (1.1)-(1.4) has a nontrivial solution provided $i_{\alpha,\beta}(B_{01})$ is odd.

The paper is organized as follows. In section 2, we give preliminary facts and provide some definitions, properties and lemmas that are needed later. The proof of Theorem 1.1 will be given in Section 3. Generally the problem (1.1)-(1.4) is not easy to investigate by variational methods. We refer to [1],[4],[5],[6],[7],[8],[9],[10],[2],[11] for some results with topological methods and other methods. For some special cases we have happily seen some results [12],[13],[14] with variational methods. In order to prove Theorem 1.1 we also use topological homotopy contiuation methods by showing for example that the possible solutions of (3.1)-(3.4) are a priori bounded in $C^1([0,1], \{t_i\}; \mathbf{R}^n)$ (see the definition in Section 3). For the special case $M_k = 0 (k = 1.2), J_j = 0 (j = 1, \cdots, p)$, Poincaré's inequality can be used as in [12]. However, for our case Poincaré's inequality can't be used directly. We transform (3.1)-(3.4) into an equivalent integral equation and then its solution is naturally divided into two parts $x = x_1 + x_2$ such that x_2 is bounded in $C^1([0,1], \{t_i\}; \mathbf{R}^n)$ and $x_1 \in H^2([0,1], \mathbf{R}^n)$ and satisfies (1.8)-(1.9). By this additional condition we can use Poincaré's inequality to show that x_1 is a priori bounded in $H^1([0,1], \mathbf{R}^n)$ and then in $C^1([0,1], \mathbf{R}^n)$, concluding the proof.

2 Some Useful Results

In this section, we present some results that are useful to the proof of the main results. For the convenience of the reader, we also present here the necessary definitions. We use $|\cdot|$ denotes the usual norm in $\mathbf{R}^n$ and $||\cdot||_1$ denotes the norm of $C^1([0,1], \mathbf{R}^n)$, we denote by $H^1([0,1], \mathbf{R}^n)$ the Sobolev space $H^1 = \{x \in L^2([0,1]; \mathbf{R}^n) : \dot{x} \in L^2([0,1]; \mathbf{R}^n)\}$ where $\dot{x}$ is weak derivative of x with the inner product

$$(x, y) = \int_0^1 x(t) \cdot y(t) dt + \int_0^1 \dot{x}(t) \cdot \dot{y}(t) dt,$$

where $x \cdot y$ denotes the inner product in $\mathbf{R}^n$. The corresponding norm is defined by $||x||_{H^1} = (x, x)^{\frac{1}{2}}, x \in H^1$.

Suppose X is a Banach space and $\Omega \subset X$ is a bounded open set. $T : \overline{\Omega} \to X$ is compact and $x - T(x)$ is not zero for all $x \in \partial\Omega$, so the Laray-Schauder degree $\deg(Id - T) \in \mathbf{Z}$ is defined.

Proposition 2.1. *(i)If* $\deg(Id - T, \Omega)$ *is not zero, then there exists* $x \in \Omega$ *such that* $x - Tx = 0$,

(ii)If K *is linear compact,* $\ker\{Id - K\} = \{0\}$ *and* $0 \in \Omega$, *then* $\deg(Id - K, \Omega) \neq 0$.

(iii)$\deg(Id - T_\lambda, \Omega)$ *is constant for* $\lambda \in [0,1]$ *provided* $x - T_\lambda$ *is not zero for any* $x \in \partial\Omega$ *and* $T_\lambda x = (1 - \lambda)T_0 x + \lambda T_1 x$ *and* $T_0, T_1 : \overline{\Omega} \to X$ *are compact.*

(iv) Assume $K : X \to X$ *is a linear compact operator,* $1 \notin \sigma(K)$ *the spectral of* K. *Let* Ω *be an open bounded subset of* X *with* $0 \in \Omega$. *Then* $\deg(Id - K, \Omega) = (-1)^\beta$ *where* $\beta = \sum_{\lambda_j > 1, \lambda_j \in \sigma(K)} \beta_j$ *and* $\beta_j = \dim \bigcup_{m=1}^{\infty} \ker(K - \lambda_j)^m$.

We also need some facts about the index. We can find the following definitions and propositions in (see [2][3]). For any $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, consider the following system

$$\ddot{x}(t) + B(t)x(t) = 0, \quad (2.1)$$
$$x(0)\cos\alpha - x'(0)\sin\alpha = 0, \quad (2.2)$$
$$x(1)\cos\beta - x'(1)\sin\beta = 0. \quad (2.3)$$

Definition 2.1 For any $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, we define

$$\nu_{\alpha,\beta}(B) = \text{the dimension of the solution space of } \ (2.1) - (2.3),$$
$$i_{\alpha,\beta}(B) = \sum_{\lambda<0} \nu_{\alpha,\beta}(B + \lambda I_n).$$

Definition 2.2 For any $B_1, B_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ with $B_1 < B_2$, we define

$$I(B_1, B_2) = \sum_{\lambda\in[0,1)} \nu_{\alpha,\beta}(\lambda B_2 + (1-\lambda)B_1).$$

We define the symmetric bilinear form

$$a(x,y) = \int_0^1 \dot{x}(t)\cdot\dot{y}(t)dt - x(1)\cdot y(1)\cot(\beta) + x(0)\cdot y(0)\cot(\alpha), \forall x, y \in H^1([0,1], \mathbf{R}^n).$$

Let $\mu \in \mathbf{R}$ such that $\nu_{\alpha,\beta}(\mu I_n) = 0$, and let $(Lx)(t) = -\ddot{x}(t) - \mu x(t)$ for $x \in H^2_{\alpha,\beta}([0,1], \mathbf{R}^n) = \{x \in H^2([0,1], \mathbf{R}^n) : x(0)\cos\alpha - x'(0)\sin\alpha = 0, x(1)\cos\beta - x'(1)\sin\beta = 0\}$, $(Qx)(t) = (B(t) - \mu I_n)x(t)$ for $x \in L^2([0,1], \mathbf{R}^n)$ and $T_{\mu,B} = Id - L^{-1}Q$. Set $U_r = \{x \in C^1([0,1], \mathbf{R}^n) | \|x\| < r\}$ for $r > 0$.

Proposition 2.2 (i) For any $B_1, B_2 \in \mathcal{L}_s(\mathbf{R}^n)$, if $B_1 \le B_2$, then $i_{\alpha,\beta}(B_1) \le i_{\alpha,\beta}(B_2), \nu_{\alpha,\beta}(B_1) + i_{\alpha,\beta}(B_1) \le \nu_{\alpha,\beta}(B_2) + i_{\alpha,\beta}(B_2)$.

(ii)For any $B_1, B_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ with $B_1 < B_2$, we have

$$I(B_1, B_2) = i_{\alpha,\beta}(B_2) - i_{\alpha,\beta}(B_1).$$

(iii) For any $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ with $\nu_{\alpha,\beta}(B) + i_{\alpha,\beta}(B) = 0$, $\phi_{a,B}(x,x) \equiv a(x,x) - \int_0^1 B(t)x \cdot xdt \ge 0, x \in H^1([0,1], \mathbf{R}^n)$ and $(\phi_{a,B}(x,x))^{\frac{1}{2}}$ is an equivalent norm on $H^1([0,1], \mathbf{R}^n)$.

(iv) There exists a constant $c_1 > 0$ such that for any $x \in H^1([0,1], \mathbf{R}^n)$, $|x(t)| \le c_1 ||x||_{H^1}$ for all $t \in [0,1]$.

(v) For $\mu \in \mathbf{R}$ and $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ with $i_{\alpha,\beta}(\mu I_n) = \nu_{\alpha,\beta}(\mu I_n) = 0$ and $B > (\mu+1)I_n$, we have

$$deg(T_{\mu,B}, U_r) = (-1)^{i_{\alpha,\beta}(B)}.$$

Proof. We only prove (v) because (i-iv) can be found in [[2],[3]]. Setting $K = L^{-1}Q$ yields

$$\sum_{\lambda>1, \lambda\in\sigma(K)} \dim\ker(K-\lambda) = \sum_{\lambda>1, \lambda\in\sigma(K)} \nu_{\alpha,\beta}(\frac{1}{\lambda}B + (1-\frac{1}{\lambda})\mu I_n)$$
$$= \sum_{\gamma\in[0,1)} \nu_{\alpha,\beta}(\gamma B + (1-\gamma)\mu I_n) = I(\mu I_n, B) = i_{\alpha,\beta}(B) - i_{\alpha,\beta}(\mu I_n) = i_{\alpha,\beta}(B).$$

By Proposition 2.1(iv) we need only to show that $\ker(K-\lambda)^2 = \ker(K-\lambda)$ for $\lambda > 1$. In fact, assume $(K-\lambda)^2 x = 0$. Then $\bar{x} \equiv (K-\lambda)x = (L^{-1} - \lambda Q^{-1})Qx \in R(L^{-1} - \lambda Q^{-1})$, $0 = (K-\lambda)\bar{x} = (L^{-1} - \lambda Q^{-1})Q\bar{x}, Q\bar{x} \in \ker(L^{-1} - \lambda Q^{-1})$. Because $L^{-1} - \lambda Q^{-1} : L^2([0,1], \mathbf{R}^n) \to L^2([0,1], \mathbf{R}^n)$ is self-adjoint, $\int_0^1 Q\bar{x}\cdot\bar{x}dt = 0$. Hence $\bar{x} = 0$. **Remark** For the index theory of convex linear Hamiltonian systems and symplectic paths
we refer to ([15] [8]).

3 Proof of the Theorem 1.1

In this section we will prove Theorem 1.1.

Proof of Theorem 1.1 Consider the homotopy problem:

$$\ddot{x} + (1-\lambda)B(t)x(t) + \lambda V'(t,x) = 0, t \in [0,1], t \neq t_j, j = 1,2,\cdots,p, \tag{3.1}$$
$$\Delta x'(t_j) = x'(t_j+0) - x'(t_j-0) = \lambda J_j(x'(t_j)), \tag{3.2}$$
$$x(0)\cos\alpha - x'(0)\sin\alpha = \lambda M_1, \tag{3.3}$$
$$x(1)\cos\beta - x'(1)\sin\beta = \lambda M_2. \tag{3.4}$$

M_k is short for $M_k(x(0), x'(0), x(1), x'(1))(k=1,2), \lambda \in (0,1)$. We first transform the problem into an equivalent integral equation. Let $\mu > 0$ be free and $f(t) \equiv (1-\lambda)(B(t)x(t) + \mu^2 x(t)) + \lambda(V'(t,x) + \mu^2 x(t)))$. Then (3.1) is equivalent to

$$\ddot{x}(t) - \mu\dot{x}(t) + \mu\dot{x}(t) - \mu^2 x(t) = -f(t)$$

Multiplying the integral factor $e^{\mu t}$ and integrating over $[0,t]$, we can get

$$\dot{x}(t) - \mu x(t) = \lambda e^{-\mu t}\sum_{t_j<t} e^{\mu t_j} J_j - e^{-\mu t}\int_0^t e^{\mu s} f(s)ds + e^{-\mu t}(\dot{x}(0) - \mu x(0)).$$

And multiplying the integral factor $e^{-\mu t}$ and integrating over $[0,t]$ again, we obtain

$$e^{-\mu t}x(t) - x(0) = \lambda\int_0^t e^{-2\mu s}\sum_{t_j<s} e^{\mu t_j} J_j ds + \int_0^t e^{-2\mu s} ds(\dot{x}(0) - \mu x(0)) - \int_0^t e^{-2\mu\tau}\int_0^\tau e^{\mu s} f(s)dsd\tau,$$

and then

$$x(t) = ch\mu t x(0) + \frac{1}{\mu} sh\mu t\dot{x}(0) + \frac{\lambda}{\mu}\sum_{t_j<t} sh\mu(t-t_j)J_j - \frac{1}{\mu}\int_0^t sh\mu(t-s)f(s)ds$$

$$\dot{x}(t) = \mu sh\mu t x(0) + ch\mu t\dot{x}(0) + \lambda\sum_{t_j<t} ch\mu(t-t_j)J_j - \int_0^t ch\mu(t-s)f(s)ds$$

Consider the boundary conditions:

$$x(1) = ch\mu x(0) + \frac{1}{\mu} sh\mu\dot{x}(0) + \frac{\lambda}{\mu}\sum_{j=1}^p sh\mu(1-t_j)J_j - \frac{1}{\mu}\int_0^1 sh\mu(1-s)f(s)ds$$

$$\dot{x}(1) = \mu sh\mu x(0) + ch\mu\dot{x}(0) + \lambda\sum_{j=1}^p ch\mu(1-t_j)J_j - \int_0^1 ch\mu(1-s)f(s)ds$$

where as usual $cha = \frac{1}{2}(e^a + e^{-a})$ and $sha = \frac{1}{2}(e^a - e^{-a})$.

Hence

$$x(0)\cos\alpha - x'(0)\sin\alpha = \lambda M_1$$

$$(ch\mu\cos\beta - \mu sh\mu\sin\beta)x(0) + (\frac{1}{\mu} sh\mu\cos\beta - ch\mu\sin\beta)x'(0) = \lambda M_2 - \cos\beta\xi_1 + \sin\beta\xi_2$$

where

$$\xi_1 = \frac{\lambda}{\mu}\sum_{j=1}^p sh\mu(1-t_j)J_j - \frac{1}{\mu}\int_0^1 sh\mu(1-s)f(s)ds,$$

$$\xi_2 = \lambda \sum_{j=1}^{p} ch\mu(1-t_j)J_j - \int_0^1 ch\mu(1-s)f(s)ds.$$

Then

$$\begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} I_n(\frac{1}{\mu}sh\mu\cos\beta - ch\mu\sin\beta) & I_n\sin\alpha \\ I_n(\mu sh\mu\sin\beta - ch\mu\cos\beta) & I_n\cos\alpha \end{pmatrix} \begin{pmatrix} \lambda M_1 \\ \lambda M_2 \end{pmatrix}$$
$$+\frac{1}{\Delta}\begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}[\sin\beta\xi_2 - \cos\beta\xi_1]$$

where we choose $\mu > 0$ large enough such that $\Delta = \frac{1}{\mu}sh\mu\cos\alpha\cos\beta + ch\mu\sin(\alpha-\beta) - \mu ch\mu\sin\alpha\sin\beta \neq 0$.

Hence

$$x(t) = \int_0^1 G(t,s)f(s)ds + \lambda(Mx)(t) \tag{3.5}$$

where

$$G(t,s) = \frac{1}{\Delta}\begin{pmatrix} I_n ch\mu t & I_n\frac{1}{\mu}sh\mu t \end{pmatrix}\begin{pmatrix} \sin\alpha I_n \\ \cos\alpha I_n \end{pmatrix}[\frac{1}{\mu}\cos\beta sh\mu(1-s) - \sin\beta ch\mu(1-s)] - \frac{1}{\mu}sh\mu(t-s)I_n$$

for $0 \le s \le t \le 1$;

$$G(t,s) = \frac{1}{\Delta}\begin{pmatrix} I_n ch\mu t & I_n\frac{1}{\mu}sh\mu t \end{pmatrix}\begin{pmatrix} \sin\alpha I_n \\ \cos\alpha I_n \end{pmatrix}[\frac{1}{\mu}\cos\beta sh\mu(1-s) - \sin\beta ch\mu(1-s)]$$

for $0 \le t \le s \le 1$;

$$\begin{aligned}(Mx)(t) \quad &= \frac{1}{\Delta}\begin{pmatrix} I_n ch\mu t & I_n\frac{1}{\mu}sh\mu t \end{pmatrix}\{\begin{pmatrix} I_n(\frac{1}{\mu}sh\mu\cos\beta - ch\mu\sin\beta) & I_n\sin\alpha \\ I_n(\mu sh\mu\sin\beta - ch\mu\cos\beta) & I_n\cos\alpha \end{pmatrix}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \\ &+ \begin{pmatrix} I_n\sin\alpha \\ I_n\cos\beta \end{pmatrix}\{\sin\beta\sum_{j=1}^{p} ch\mu(1-t_j)J_j(x'(t_j)) - \cos\beta\frac{1}{\mu}\sum_{j=1}^{p} sh\mu(1-t_j)J_j(x'(t_j))\} \\ &+ \frac{1}{\mu}\sum_{t_j<t}^{p} sh\mu(t-t_j)J_j(x'(t_j))\}.\end{aligned} \tag{3.6}$$

Then $x(t)$ is a solution of (3.1)-(3.4) if and only if $x(t)$ is a solution of the integral equation (3.5). Let $Y = C^1([0,1],\{t_i\};\mathbf{R}^n) = \{x : [0,1] \to \mathbf{R}^n | x'(t)$ exists and continuous for $t \neq t_i, x'(t_i \pm 0)$ exists and $x(t_i) = x(t_i-0), x'(t_i) = x'(t_i-0)$ for $i = 1,2,\cdots,p\}$with the norm $||x|| = \sup_{t\in[0,1]}\{|x'(t)|,|x(t)|\}$. The operator $M : Y \to Y$ defined by (3.6) is compact.

Set $(T_\lambda x)(t) = f(t) = (1-\lambda)(B(t)x(t) + \mu^2 x(t)) + \lambda(V'(t,x) + \mu^2 x(t))$ for $x \in Y$. $(Fu)(t) = \int_0^1 G(t,s)u(s)ds$ for $u \in L^2([0,1],\mathbf{R}^n)$ and $(Ax)(t) = -\ddot{x}(t) + \mu^2 x(t)$ for $x \in H^2_{\alpha,\beta}([0,1],\mathbf{R}^n)$. Then $T_\lambda : Y \to L^2([0,1],\mathbf{R}^n)$ are continuous and $F = A^{-1} : L^2([0,1],\mathbf{R}^n) \to Y$ is compact. Hence (3.1)-(3.4) can be viewed as the operator equation

$$x = A^{-1}T_\lambda x + \lambda Mx = (1-\lambda)FT_0 x + \lambda(FT_1 + M)x. \tag{3.7}$$

So in order to use topological degree to investigate (3.7) it suffices to show the possible solutions of (3.1)-(3.4)are a priori bounded in Y. Let $x(t) = \int_0^1 G(t,s)f(s)ds + \lambda(Mx)(t) = x_1(t) + x_2(t)$, where $x_1(t) = \int_0^1 G(t,s)f(s)ds = FT_\lambda x \in H^2_{\alpha,\beta}([0,1],\mathbf{R}^n), x_2(t) = \lambda(Mx)(t)$. x_2 is bounded in Y because of the boundedness of $J_j(j = 1,2,\cdots,p)$ and $M_k(k = 1,2)$.

Next we show $||x_1||$ is also bounded, we only need to show $||x_1||_{H^1}$ is bounded.

By (V_1), there is a $c_2 > 0$ such that

$$\int_0^1 V'(t,x)\cdot x(t)dt \le \int_0^1 B(t)x(t)\cdot x(t)dt + c_2, \forall (t,x)\in[0,1]\times\mathbf{R}^n.$$

And $x(t)$ satisfies (3.1)-(3.4), we have

$$\begin{aligned}
0 &= \int_0^1 \ddot{x}(t)\cdot x(t)dt + \int_0^1 (1-\lambda)B(t)x(t)\cdot x(t)dt + \lambda\int_0^1 V'(t,x)\cdot x(t)dt \\
&\le x(1)\dot{x}(1) - x(0)\dot{x}(0) - \sum_{j=1}^p x(t_j)J_j(x(t_j)) - \int_0^1 |\dot{x}(t)|^2 dt + \int_0^1 B(t)x(t)\cdot x(t)dt + \lambda c_2 \\
&\le |x(1)|^2\cot\beta - |x(0)|^2\cot\alpha - \frac{\lambda}{\sin\beta}x(1)\cdot M_2 + \frac{\lambda}{\sin\alpha}x(0)\cdot M_1 - \lambda\sum_{j=1}^p x(t_j)\cdot J_j(x'(t_j)) \\
&- \int_0^1 |\dot{x}(t)|^2 dt + \int_0^1 B(t)x(t)\cdot x(t)dt + \lambda c_2 \\
&\le |x_1(1)|^2\cot\beta - |x_1(0)|^2\cot\alpha - \int_0^1 |\dot{x}_1(t)|^2 dt + c_3|x_1(1)| + c_3|x_1(0)| + \int_0^1 B(t)x_1\cdot x_1 dt \\
&+ c_3\sum_{j=1}^p x_1(t_j)) + c_4 = -\phi_{a,B}(x_1,x_1) + c_3(|x_1(1)| + x_1(0) + \sum_{j=1}^p |x_1(t_j)|) + c_4
\end{aligned}$$

where c_3 and c_4 are constants.

By Proposition 2.2(iii)(iv), there exists $c_5 > 0$

$$||x_1||_{H^1}^2 \le c_5(||x_1||_{H^1} + 1)$$

Hence $||x_1||_{H^1}$ is bounded, we obtain $||x||$ is bounded.

The above argument shows that there is $R > 0$ such that $||x|| > R, \ddot{x}+(1-\lambda)B(t)x(t)+\lambda V'(t,x) \ne 0$. Set $N_\lambda : Y \to Y$ by $(N_\lambda x)(t) = (1-\lambda)(A^{-1}T_0x)(t) + \lambda((A^{-1}T_1 + M)x)(t)$; then $\deg(Id - N_\lambda, U_R)$ is well defined. $\ker(Id - N_0) = \ker(Id - A^{-1}T_0) = \{0\}$ because of $\nu_{\alpha,\beta}(B) = 0$ and Proposition 2.1(ii)(iii) lead to $\deg(Id - N_0, U_R) \ne 0$ and $\deg(Id - N_1, U_R) = \deg(Id - N_0, U_R) \ne 0$. Then (1.1)-(1.4) has one solution via Proposition 2.1(i).

Now we further assume $(V_2), (M), (J)$ hold, we prove that the following problem:

$$x - (1-\lambda)(A^{-1}T_2x) - \lambda(A^{-1}T_1 + M)x = 0$$

has no solution x satisfying $0 < ||x||_1 \le r_0$ where $T_2 : Y \to Y$ by $(T_2x)(t) = B_{01}(t)x(t) + \mu^2 x(t)$ and $r_0 > 0$ is small enough and recall that $T_1 = V'(t,x) + \mu^2 x(t)$. If not, there exist $\{x_k\}_{k=1}^\infty \subset Y$ such that $||x_k|| \to 0$ and $\{\lambda_k\}_{k=1}^\infty \subset (0,1)$ such that

$$x_k - (1-\lambda_k)(A^{-1}T_2x_k) - \lambda_k(A^{-1}T_1 + M)x_k = 0 \tag{3.8}$$

Set $\bar{B}_k(t) = (1-\lambda_k)(B_{01}(t) + \mu^2 I_n) + \lambda_k(B_0(t,x_k) + \mu^2 I_n)$ and $y_k = \frac{x_k}{||x_k||}$, then (3.8) turns to

$$y_k - (A^{-1}\bar{B}_k y_k) - \lambda_k\frac{Mx_k}{||x_k||} = 0 \tag{3.9}$$

By (M) and (J), we know $\frac{Mx_k}{||x_k||} \to 0$ in Y. Since $||y_k|| = 1, \{\bar{B}_k y_k\}$ is bounded and weakly convergent in $L^2([0,1],\mathbf{R}^n)$ and hence $y_k \to y_0$ in Y via (3.9) by going to subsequences if necessary.

Set$\widetilde{B}_k(t) = (1-\lambda_k)B_{01}(t) + \lambda_k B_0(t,x)$; then $B_{01}(t) \le \widetilde{B}_k(t) \le B_{02}(t)$ and we claim that there is a $D_0 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ satisfying $B_{01} \le D_0 \le B_{02}$ such that

$$\widetilde{B}_k y_k \rightharpoonup D_0 y_0, \tag{3.10}$$

in $L^2([0,1], \mathbf{R}^n)$. In fact, setting $B_{01} = (b_{ij}^{(01)}(t))_{n\times n}, B_{02} = (b_{ij}^{(02)}(t))_{n\times n}$ and $\widetilde{B}_k(t) = (\widetilde{b}_{ij}^k(t))_{n\times n}$,we have

$$b_{ii}^{(01)} \le \widetilde{b}_{ii}^k \le b_{ii}^{(02)}, \forall i = 1, 2, \cdots, n$$

$$2b_{ij}^{(01)} + b_{ii}^{(01)} + b_{ij}^{(01)} \le 2\widetilde{b}_{ij}^k + \widetilde{b}_{ii}^k + \widetilde{b}_{ij}^k \le 2b_{ij}^{(02)} + b_{ii}^{(02)} + b_{ij}^{(02)}, \forall j \ne i.$$

We can get $\widetilde{b}_{ij}^k \rightharpoonup b_{ij}$ in $L^2([0,1], \mathbf{R})$ by going to subsequences if necessary. Then setting $D_0(t) = (b_{ij}(t))_{n\times n}$ leads to the results. Since $A^{-1} : L^2([0,1], \mathbf{R}^n) \to Y$ is compact and $\bar{B}_k y_k = \widetilde{B}_k y_k + \mu^2 y_k$, we can get $A^{-1}\bar{B}_k y_k \to A^{-1}(D_0 y_0 + \mu^2 y_0)$ in Y via (3.10).

Taking limit in (3.9) yields

$$y_0 - A^{-1}(D_0 y_0 + \mu^2 y_0) = 0$$

and $||y_0|| = 1$. This means that $y = y_0$ is a nontrivial solution of the following system

$$\begin{aligned} &\ddot{y}(t) + D_0 y(t) = 0, \\ &y(0)\cos\alpha - y'(0)\sin\alpha = 0, \\ &y(1)\cos\beta - y'(1)\sin\beta = 0. \end{aligned}$$

Because $i_{\alpha,\beta}(B_{01}) = i_{\alpha,\beta}(B_{02}), \nu_{\alpha,\beta}(B_{02}) = 0$, by Proposition 2.1, $\nu_{\alpha,\beta}(D_0) = 0$, a contradiction.

Hence Proposition 2.3(iv) yields $deg(Id - N_1, U_R \backslash \bar{U}_{r_0}) = deg(Id - N_1, U_R) - deg(Id - N_1, \bar{U}_{r_0}) = (-1)^{i_{\alpha,\beta}(B)} - (-1)^{i_{\alpha,\beta}(B_{01})} = 1 - (-1)^{i_{\alpha,\beta}(B_{01})} = 2 \ne 0$ because $i_{\alpha,\beta}(B_{01})$ is odd. Therefore, (3.1)-(3.4) has one solution x with $||x||_1 \in (r_0, R)$.

4 Conclusions

Thoughout this paper, we obtain some results on the nonlinear Sturm-Liouville boundary value problems for second order Hamiltonian systems with impulsive effects. Compared the results obtained in [1] and [2], we generalized some previous results.

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Competing Interests

Author has declared that no competing interests exist.

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