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Completion of Weakly Sign Symmetric P_O-Matrix Problem for 5 × 5 Matrices Specifying Digraphs of Order 5 with UP to 5 Arcs

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

An n × n matrix is a weakly sign symmetric matrix if the off-diagonal elements have the property that if the entry in row i and column j is non-zero, then the entry in row j and column i must have same sign or zero. A digraph D has a Wss P_o -matrix completion if every partial weakly sign symmetric P_o -matrix that describes D can be extended to a complete weakly sign symmetric P_o -matrix. This paper investigates the problem of completing weakly sign symmetric P_o -matrices. It demonstrates that partial matrices representing all directed graphs of order 5 with edge strengths from 0 to 5 can indeed be completed to a weakly sign symmetric P_o -matrix. Moreover, we established digraph characteristics that the partial Wss P_o- matrices specifying digraphs of order 5 with up to 5 arcs which have a clique and are cyclic or acyclic have zero completion into a Wss P_o-

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matrix. Insights gained from this class of matrix could be applied to fill gaps in data surveys, and business analytics by analysing complex relationships, allocating resources, network modelling, optimizing processes and managing risks.

Keywords: *Cyclic digraphs; acyclic digraphs; clique digraphs; Matrix completion; Digraph; weakly sign symmetric P_o-matrix.*

1 Introduction

Completion of Wss P_o- matrix for 4×4 matrices has been explored by [1,2] using digraphs of order 4 with 4 arcs. However, the case of digraphs of order 5×5 has not been investigated. In this study we determine Wss P_o- matrix completion of digraphs of order 5 with up to 5 arcs. A P_o-matrix A is classified as a weakly sign symmetric P_0-matrix if $a_{ij}\, a_{ji \geq}$ 0 for all i and j [3]. A sub matrix that has no unspecified entry is said to be fully specified according to [4,5,6]. Let α be a subset of N = 1, 2, …., n. the principal submatrix obtained by deleting all columns and rows not in α from A is denoted as A (α). Similarly, P(α) represents the principal sub pattern obtained from the pattern P by removing all positions where the first and second coordinates are not in α. A principal minor of A is the determinant of a principal sub-matrix [7]. A square matrix A is n × n, with equal rows and columns. A partial matrix is an array of numbers with some specified entries while others unspecified. A partial Po-matrix has non-negative principal submatrices. A fully specified submatrix of A has all entries defined [8,9].

2 Preliminaries

Basic concepts in linear algebra, group theory and graph theory that are commonly used in Wss P_o- matrix completion problems are defined in the section below.

Definition: 2.1 Graphs and digraphs is used in matrix completion of various classes of matrices. A graph, denoted as G = (V_G, E_G) comprises a finite non-empty set of positive integers V_G, where vertices are the members, along with a set of unordered pairs {u, v} of vertices called edges. A null graph is a graph devoid of edges [10,5].

Definition: 2.2 A digraph D = (V_D, E_D) is like a graph G but includes directed edges (u, v) where u is the start vertex and v is the end vertex. A digraph with cycles is termed a cycle digraph; without cycles, it's acyclic. The order of a digraph is its vertex count, and its size is the number of arcs. A clique in a digraph contains all possible arcs between vertices [7].

Definition: 2.3 Matrix completion involves identifying feasible positions in matrices using patterns. Patterns like symmetric pairs (i, j) and (j, i) are crucial for n × n submatrices, focusing on diagonal elements. Symmetric properties in weakly sign symmetric Po-matrices facilitate problem-solving in completing matrices, utilizing specified entries corresponding to outlined patterns [11].

Definition: 2.4 A pattern D is considered a permutation similar to a pattern B if there exists a permutation ϕ of {1,2…., n} such that B is formed by mapping the pairs (i, j) in D to ϕ(i), ϕ (j)) [4].

Lemma: 2.5 Weakly sign symmetric Po-matrices exhibit closure under similarity transformations by permutations.

Weakly sign symmetric P_o-matrices, due to their closure under permutation similarity, allow digraph relabeling. If a P_o-matrix A has non-negative elements a_{ij} and there exists a permutation matrix P such that PAP^T is sign symmetric, then A is considered a weakly sign symmetric P_o-matrix [1,2].

Definition: 2.6 Let A be a Wss P_o- matrix. Then

(i) If P is a permutation matrix, then PAP^T is a Wss P_o- matrix
(ii) Any principal sub-matrix of A is Wss P_o- matrix. The set of Wss P_o- matrices is closed under permutation similarity and left and right diagonal multiplication.

Theorem 2.7 A permutation matrix P is obtained by interchanging row on the identity matrix. The permutation matrix P is given by PAP^T. This is represented on the digraph by relabeling the vertices of the digraph [12].

Proposition: 2.8 Every n × n partial Wss P_o- matrix with all unspecified off diagonal entries has Wss P_o-matrix completion.

Theorem: 2.9 Let $A = [a_{ij}]$ be a partial Wss P_o-matrix and $\alpha = \{i: a_{ij}\}$ is specified. If the principal partial submatrix A (α) Of A has a Wss P_o-matrix completion, then A has Wss P_o-completion [13].

3. Analysis of 5×5 Matrices Specifying Digraphs with 5 Vertices

In the process of constructing a partial Wss P_o- matrix, we use a digraph where vertices correspond to diagonal entries $d_{ii.}$ This class of matrices allows 0 as an entry, and by definition, diagonal entries are non-negative (≥ 0). Digraphs are utilized to create partial Wss P_o- matrices, where specific entries were known denoted by a_{ij} corresponding to existing arcs in the digraph, and other entries are unspecified denoted by X_{ij} representing missing arcs. digraphs with 5 vertices and 0 to 5 arcs are considered. In this paper we take our d_{ii} and a_{ij} to be 1 respectively.

Digraph D of order 5 without an arc; consider the diagraph D= {(1,1) (2,2), (3,3), (4,4), (5,5)} with 5 vertices and no arc given by:

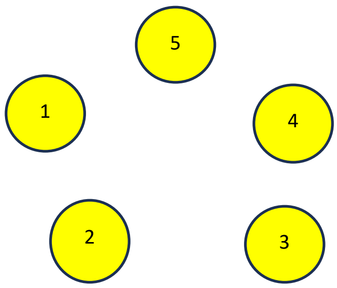

Fig. 1. Digraph D of order 5

The partial matrix that specifies the digraph D is

$$A = \begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$$

By definition of partial Wss P_o- matrix $d_1 \geq 0$, $d_2 \geq 0$, $d_3 \geq 0$, $d_4 \geq 0$, $d_5 \geq 0$.

For d_{ii} =1, a_{ij} =1, then the partial matrix becomes $A = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & 1 & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & 1 & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & 1 & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix}$. Next we compute the principal minors.

Det (1,2) = $d_{11}d_{22} - x_{12}x_{21.}$ Setting the unspecified entry to zero, and d_{ii} =1, i.e., Det (1,2) = 1– 0 = 1 > 0. Similarly,
Det (1,3) = $d_{11}d_{33} - x_{13}x_{31,}$ Det (1,4) = $d_{11}d_{44} - x_{14}x_{41}$
Det (1,5) = $d_{11}d_{55} - x_{15}x_{51,}$ Det (2,3) = $d_{22}d_{33} - x_{23}x_{32,}$ Det (2,4) = $d_{22}\, d_{44} - x_{24}x_{42}$
Det (2,5) = $d_{22}\, d_{55} - x_{25}x_{52,}$ Det (3,4) = $d_{33}\, d_{44} - x_{34}x_{43,}$ Det (3,5) = $d_{33}\, d_{55} - x_{35}x_{53}$
Det (4 5) = $d_{44}\, d_{55} - x_{45}x_{54}$.

Table 1. Determinants of 2×2 sub-matrices

Principal submatrix	Principal minor
A (1,2)	Det A (1,2) = $d_{11}d_{22} \geq 0$.
A (1,3)	Det A (1,3) = $d_{11}d_{33} \geq 0$.
A (1,4)	Det A (1,4) = $d_{11}d_{4\,4} \geq 0$.
A (1,5)	Det A (1,5) = $d_{11}d_{55} \geq 0$.
A (2,3)	Det A (2,3) =$d_{22}d_{33} \geq 0$.
A (2,4)	Det A (2,4) = $d_{22}d_{44} \geq 0$.
A (2,5)	Det A (2,5) = $d_{22}d_{55} \geq 0$.
A (3,4)	Det A (3,4) = $d_{33}d_{44} \geq 0$.
A (3,5)	Det A (3,5) = $d_{33}d_{55} \geq 0$.
A (4,5)	Det A (4,5) = $d_{44}d_{55} \geq 0$.

By definition of completion, determinants of 2 × 2 submatrices are obtained. i.e., det A (1,3) = $d_{11}d_{33} - x_{13}x_{31} \geq 0$, since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero then; det A (1,3) = 1– 0 = 1 > 0. Therefore, all the determinants of 2 × 2 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Table 2. Determinants of 3×3 sub-matrices

Principal submatrix	Principal minor
A (1,2,3)	Det A (1,2,3) = $d_{11}d_{22}d_{33} \geq 0$.
A (1,2,4)	Det A (1,2,4) = $d_{11}d_{22}d_{44} \geq 0$.
A (1,2,5)	Det A (1,2,5) = $d_{11}d_{22}d_{55} \geq 0$.
A (1,3,4)	Det A (1,3,4) = $d_{11}d_{33}d_{44} \geq 0$.
A (1,3,5)	Det A (1,3,5) = $d_{11}d_{33}d_{55} \geq 0$.
A (1,4,5)	Det A (1,4,5) = $d_{11}d_{44}d_{55} \geq 0$.
A (2,3,4)	Det A = (2,3,4) $d_{22}d_{33}d_{44} \geq 0$.
A (2,3,5)	Det A (2,3,5) =$d_{22}d_{33}d_{55} \geq 0$.
A (2,4,5)	Det A (2,4,5) = $d_{22}d_{44}d_{55} \geq 0$.
A (3,4,5)	Det A (3,4,5) = $d_{33}d_{44}d_{55} \geq 0$.

Determinants of 3 × 3 submatrices are then obtained. i.e., det A (1,2,3) = $d_{11}(d_{22}d_{33} - x_{23}x_{32}) - x_{12}(x_{21}d_{33} - x_{23}x_{31}) + x_{13}(x_{21}x_{32} - d_{22}x_{31}) \geq 0$., since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero then; det A (1,2,3) = 1– 0 = 1 > 0. Therefore, all the determinants of 3 × 3 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Table 3. Determinants of 4×4 sub-matrices

Principal sub-matrix	Principal minor
A (1,2,3,4)	A (1,2,3,4) = $d_{11}d_{22}d_{33}d_{44} \geq 0$.
A (1,2,3,5)	A (1,2,3,5) = $d_{11}d_{22}d_{33}d_{55} \geq 0$.
A (1,2,4,5)	A (1,2,4,5) = $d_{11}d_{22}d_{44}d_{55} \geq 0$.
A (1,3,4,5)	A (1,3,4,5) = $d_{11}d_{33}d_{44}d_{55} \geq 0$.
A (2,3,4,5)	A (2,3,4,5) = $d_{22}d_{33}d_{44}d_{55} \geq 0$.

Det (A) = $d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$. Therefore det (A) = 1 > 0.

Determinants of 4 × 4 submatrices are obtained. i.e., det A (1,2,3,4) = Det (1,2,3,4) = $d_{11}[d_{22}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(d_{44}x_{32} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})]$

$$- x_{12}[x_{21}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(d_{44}x_{31} - x_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})]$$
$$+ x_{13}[x_{21}(x_{32}d_{44} - x_{34}x_{42}) - d_{22}(d_{44}x_{31} - x_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})]$$
$$- x_{14}[x_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{43}x_{31} - d_{33}x_{41}) + x_{23}(x_{31}x_{42} - x_{32}x_{41})] \geq 0.$$

since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero then; det A (1,2,3,4) = 1– 0 = 1 > 0. Therefore, all the determinants of 4 × 4 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Digraph D of order 5 and 1 arc: Consider the digraph D= {(1,1) (2,2), (2,5), (3,3), (4,4), (5,5)} with 5 vertices and one arc given by:

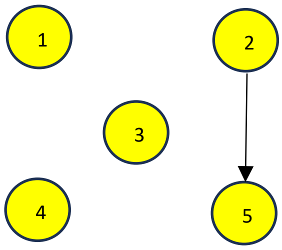

Fig. 2. **Digraph D of order 5 and 1 arc**

The Partial matrix that specifies the digraph D is

$$A = \begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & a_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$$

By definition of partial Wss P_o- matrix $d_1 \geq 0$, $d_2 \geq 0$, $d_3 \geq 0$, $d_4 \geq 0$, $d_5 \geq 0$.

For d_{ii} =1, a_{ij} =1, then the partial matrix becomes A= $\begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & 1 & x_{23} & x_{24} & 1 \\ x_{31} & x_{32} & 1 & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & 1 & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix}$. Next, we compute the principal minors.

Det (1,2) = $d_{11}\, d_{22} - x_{12}\, x_{21}$. Setting the unspecified entry to zero, and d_{ii} =1, a_{ij} =1, i.e., Det (1,2) = 1– 0 = 1 > 0.
Det (2,5) = $d_{22}\, d_{55} – a_{25}\, x_{52}$, i.e., $a_{25}\, x_{52} = 1.0 = 0$. Similarly,
Det (1,3) = $d_{11}\, d_{33} - x_{13}\, x_{31}$, Det (1,4) = $d_{11}\, d_{44} - x_{14}\, x_{41}$
Det (1,5) = $d_{11}\, d_{55} - x_{15}\, x_{51}$, Det (2,3) = $d_{22}\, d_{33} – x_{23}\, x_{32}$, Det (2,4) = $d_{22}\, d_{44} – x_{24}\, x_{42}$
Det (2,5) = $d_{22}\, d_{55} – a_{25}\, x_{52}$, Det (3,4) = $d_{33}\, d_{44} – x_{34}\, x_{43}$, Det (3,5) = $d_{33}\, d_{55} – x_{35}\, x_{53}$
Det (4 5) = $d_{44}\, d_{55} – x_{45}\, x_{54}$.

Table 4. Determinants of 2×2 sub-matrices

Principal submatrix	Principal minor
A (1,2)	Det A (1,2) = $d_{11}d_{22} \geq 0$.
A (1,3)	Det A (1,3) = $d_{11}d_{33} \geq 0$.
A (1,4)	Det A (1,4) = $d_{11}d_{44} \geq 0$.
A (1,5)	Det A (1,5) = $d_{11}d_{55} \geq 0$.
A (2,3)	Det A (2,3) =$d_{22}d_{33} \geq 0$.
A (2,4)	Det A (2,4) = $d_{22}d_{44} \geq 0$.
A (2,5)	Det A (2,5) = $d_{22}d_{55} \geq 0$.
A (3,4)	Det A (3,4) = $d_{33}d_{44} \geq 0$.
A (3,5)	Det A (3,5) = $d_{33}d_{55} \geq 0$.
A (4,5)	Det A (4,5) = $d_{44}d_{55} \geq 0$.

By definition of completion, determinants of 2 × 2 submatrices are obtained. i.e. det (2,5) = $d_{22}\, d_{55} – a_{25}\, x_{52} \geq 0$, since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij}

=1 then; det A (1,3) = 1– 0 = 1 > 0. Therefore, all the determinants of 2 × 2 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Table 5. Determinants of 3×3 sub-matrices

Principal submatrix	Principal minor
A (1,2,3)	Det A (1,2,3) = $d_{11}d_{22}d_{33} \geq 0$.
A (1,2,4)	Det A (1,2,4) = $d_{11}d_{22}d_{44} \geq 0$.
A (1,2,5)	Det A (1,2,5) = $d_{11}d_{22}d_{55} \geq 0$.
A (1,3,4)	Det A (1,3,4) = $d_{11}d_{33}d_{44} \geq 0$.
A (1,3,5)	Det A (1,3,5) = $d_{11}d_{33}d_{55} \geq 0$.
A (1,4,5)	Det A (1,4,5) = $d_{11}d_{44}d_{55} \geq 0$.
A (2,3,4)	Det A = (2,3,4) $d_{22}d_{33}d_{44} \geq 0$.
A (2,3,5)	Det A (2,3,5) =$d_{22}d_{33}d_{55} \geq 0$.
A (2,4,5)	Det A (2,4,5) = $d_{22}d_{44}d_{55} \geq 0$.
A (3,4,5)	Det A (3,4,5) = $d_{33}d_{44}d_{55} \geq 0$.

Determinants of 3 × 3 submatrices are obtained. i.e., det A (1,2,3) = d_{11} $(d_{22}d_{33} - x_{23}\, x_{32})$ - x_{12} $(x_{21}\, d_{33} - x_{23}\, x_{31})$ + $x_{13}\,(x_{21}\, x_{32} - d_{22}x_{31}) \geq 0$., since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij} =1 then; det A (1,2,3) = 1– 0 = 1 > 0. Therefore, all the determinants of 3 × 3 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Table 6. Determinants of 4×4 sub-matrices

Principal sub-matrix	Principal minor
A (1,2,3,4)	A (1,2,3,4) = $d_{11}d_{22}d_{33}d_{44} \geq 0$.
A (1,2,3,5)	A (1,2,3,5) = $d_{11}d_{22}d_{33}d_{55} \geq 0$.
A (1,2,4,5)	A (1,2,4,5) = $d_{11}d_{22}d_{44}d_{55} \geq 0$.
A (1,3,4,5)	A (1,3,4,5) = $d_{11}d_{33}d_{44}d_{55} \geq 0$.
A (2,3,4,5)	A (2,3,4,5) = $d_{22}d_{33}d_{44}d_{55} \geq 0$.

Det (A) = $d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$. Therefore det (A) = 1 > 0.

Determinants of 4 × 4 submatrices are obtained. i.e., det A (1,2,3,4) = Det (1,2,3,4) = $d_{11}[d_{22}\,(d_{33}\, d_{44} - x_{34}\, x_{43})$ - $x_{23}\,(d_{44}\, x_{32} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})]$

$- x_{12}\,[x_{21}\,(d_{33}\, d_{44} - x_{34}\, x_{43})$ - $x_{23}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{43} - d_{33}\, x_{41})]$
$+ x_{13}\,[x_{21}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) - d_{22}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{42} - x_{32}\, x_{41})]$
- $x_{14}\,[x_{21}\,(x_{32}\, x_{43} - d_{33}\, x_{42})$ - $d_{22}(x_{43}\, x_{31} - d_{33}\, x_{41}) + x_{23}\,(x_{31}\, x_{42} - x_{32}\, x_{41})] \geq 0$. since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij} =1 then; det A (1,2,3,4) = 1– 0 + 0 – 0 = 1 > 0. Therefore, all the determinants of 4 × 4 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Digraph D of order 5 and 2 arcs: consider a digraph D = {(1,1), (2,2), (3,3), (3,4), (4,3) (4,4), (5,5)} with 5 vertices and 2 arcs given by:

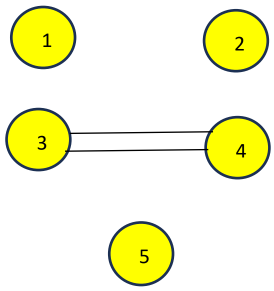

Fig. 3. Digraph D of order 5 and 2 arcs

The Partial matrix that specifies the digraph D is

$$A = \begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & a_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}.$$

By definition of partial Wss P_o- matrix $d_1 \geq 0$, $d_2 \geq 0$, $d_3 \geq 0$, $d_4 \geq 0$, $d_5 \geq 0$.

For $d_{ii} = 1$, $a_{ij} = 1$, then the partial matrix becomes $A = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & 1 & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & 1 & 1 & x_{35} \\ x_{41} & x_{42} & 1 & 1 & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix}$**.** Next, we compute the principal minors.

Det (1,2) = $d_{11}d_{22} - x_{12}x_{21}$, Setting the unspecified entry to zero, i.e., $x_{12}x_{21} = 0$, Det (3,4) = $d_{33}\, d_{44} - a_{34}\, a_{43} = 1\text{-}1 \geq 0$. Similarly;
Det (1,3) = $d_{11}\, d_{33} - x_{13}\, x_{31}$, Det (1,4) = $d_{11}\, d_{44} - x_{14}\, x_{41}$
Det (1,5) = $d_{11}\, d_{55} - x_{15}\, x_{51}$, Det (2,3) = $d_{22}\, d_{33} - x_{23}\, x_{32}$,
Det (2,4) = $d_{22}\, d_{44} - x_{24}\, x_{42}$, Det (2,5) = $d_{22}\, d_{55} - x_{25}\, x_{52}$,
Det (3,5) = $d_{33}\, d_{55} - x_{35}\, x_{53}$, Det (4 5) = $d_{44}\, d_{55} - x_{45}\, x_{54}$

Table 7. Determinants of 2×2 sub-matrices

Principal submatrix	Principal minor
A (1,2)	Det A (1,2) = $d_{11}d_{22} \geq 0$.
A (1,3)	Det A (1,3) = $d_{11}d_{33} \geq 0$.
A (1,4)	Det A (1,4) = $d_{11}d_{44} \geq 0$.
A (1,5)	Det A (1,5) = $d_{11}d_{55} \geq 0$.
A (2,3)	Det A (2,3) =$d_{22}d_{33} \geq 0$.
A (2,4)	Det A (2,4) = $d_{22}d_{44} \geq 0$.
A (2,5)	Det A (2,5) = $d_{22}d_{55} \geq 0$.
A (3,4)	Det A (3,4) = $(d_{33}d_{44}) - (a_{34}a_{43}) \geq 0$
A (3,5)	Det A (3,5) = $d_{33}d_{55} \geq 0$.
A (4,5)	Det A (4,5) = $d_{44}d_{55} \geq 0$.

By definition of completion, determinants of 2 × 2 submatrices are obtained. i.e. Det (1,2) = $d_{11}d_{22} - x_{12}x_{21} \geq 0$, Det (3,4) = $d_{33}\, d_{44} - a_{34}\, a_{43} \geq 0$. Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and $a_{ij} = 1$ then; det A (1,2) = 1– 0 = 1 > 0, Det (3,4) = 1-1 ≥ 0. Therefore, all the determinants of 2 × 2 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Table 8. Determinants of 3×3 sub-matrices

Principal submatrix	Principal minor
A (1,2,3)	Det A (1,2,3) = $d_{11}d_{22}d_{33} \geq 0$.
A (1,2,4)	Det A (1,2,4) = $d_{11}d_{22}d_{44} \geq 0$.
A (1,2,5)	Det A (1,2,5) = $d_{11}d_{22}d_{55} \geq 0$.
A (1,3,4)	Det A (1,3,4) = $d_{11}d_{33}d_{44} \geq 0$.
A (1,3,5)	Det A (1,3,5) = $d_{11}d_{33}d_{55} \geq 0$.
A (1,4,5)	Det A (1,4,5) = $d_{11}d_{44}d_{55} \geq 0$.
A (2,3,4)	Det A = (2,3,4) $d_{22}d_{33}d_{44} \geq 0$.
A (2,3,5)	Det A (2,3,5) =$d_{22}d_{33}d_{55} \geq 0$.
A (2,4,5)	Det A (2,4,5) = $d_{22}d_{44}d_{55} \geq 0$.
A (3,4,5)	Det A (3,4,5) = $d_{55}(d_{33}d_{44}) - (a_{34}a_{43}) \geq 0$

Determinants of 3 × 3 submatrices are obtained. i.e., Det (3,4,5) = d_{33} (d_{44} d_{55} – x_{45} x_{54}) – a_{34} (a_{43} d_{55} – x_{45} x_{53}) + x_{35} (a_{43} x_{54} – d_{44} x_{53}) ≥ 0. Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij} =1 then; det A (3,4,5) = 1– 1 = 0. Therefore, all the determinants of 3 × 3 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Table 9. Determinants of 4×4 sub-matrices

Principal sub-matrix	Principal minor
A (1,2,3,4)	Det (1,2,3,4) = $d_{11}d_{22}d_{33}d_{44} \geq 0$.
A (1,2,3,5)	Det (1,2,3,5) = $d_{11}d_{22}d_{33}d_{55} \geq 0$.
A (1,2,4,5)	Det (1,2,4,5) = $d_{11}d_{22}d_{44}d_{55} \geq 0$.
A (1,3,4,5)	Det (1,3,4,5) =$d_{11}d_{55}(d_{33}d_{44}) - (a_{34}a_{43}) \geq 0$.
A (2,3,4,5)	Det (2,3,4,5) = $d_{22}d_{33}d_{44}d_{55} \geq 0$.

Det (A) = $d_{11}d_{22}d_{33}d_{44}d_{55}) - (a_{34}a_{43}) \geq 0$. therefore Det (A) = 1– 1 = 0.

Determinants of 4 × 4 submatrices are obtained. i.e., Det (1,3,4,5) = d_{11} [d_{33} (d_{44} d_{55} – x_{45} x_{54}) – a_{34} (d_5 a_{43} – x_{45} x_{53}) + x_{35} (a_{43} x_{54} – d_{44} x_{53})]

– x_{13} [x_{31} (d_{44} d_{55} – x_{45} x_{54}) – a_{34} (d_5 x_{41} – x_{45} x_{51}) + x_{35} (x_{41} x_{54} – d_{44} x_{51})]
+ x_{14} [x_{31} (d_{55} a_{43} – x_{45} x_{53}) – d_{33} (d_5 x_{41} – x_{45} x_{51}) + x_{35} (x_{41} x_{53} – a_{43} x_{51})]
- x_{15} [x_{31} (a_{43} x_{54} – d_{44} x_{53}) – d_{33} (x_{41} x_{54} – d_{44} x_{51}) + a_{34} (x_{41} x_{53} – x_{51} a_{43})] ≥ 0. Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij} = 1 then; det A (1,3,4,5) = 0 – 0 + 0 – 0 = 0. Therefore, all the determinants of 4 × 4 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Digraph D of order 5 and 3 arcs; consider a digraph D = {(1,1), (1,2), (1,3), (2,1), (2,2), (3,3), (4,4), (5,5)} with 5 vertices and 3 arcs given by:

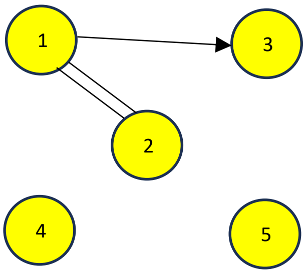

Fig. 4. Digraph D of order 5 and 3 arcs

The partial matrix that specifies the digraph D is $\boldsymbol{A} = \begin{pmatrix} d_{11} & a_{12} & a_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

By definition of partial Wss P_o- matrix $d_1 \geq 0$, $d_2 \geq 0$, $d_3 \geq 0$, $d_4 \geq 0$, $d_5 \geq 0$.

For d_{ii} = 1, a_{ij} = 1, then the partial matrix becomes A = $\begin{pmatrix} 1 & 1 & 1 & x_{14} & x_{15} \\ 1 & 1 & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & 1 & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & 1 & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix}$. Next, we compute the principal minors.

Det (1,2) = $d_{11}d_{22} - a_{12}a_{21}$. Setting unspecified entries to zero, d_{ii} =1, a_{ij} =1, then we have; 1-1 ≥ 0. Det (1,3) = $d_{11}d_{33} - a_{13}x_{31}$ = 1– x_{31} = 1– 0 = 1 ≥ 0. Similarly
Det (1,2) = $d_{11}d_{22} - a_{12}a_{21}$, Det (1,4) = $d_{11}d_{44} - x_{14}x_{41}$,

Det (1,5) = $d_{11}d_{55}$ - $x_{15}x_{51,}$ Det (2,3) = $d_{22}d_{33} – x_{23}x_{32,}$ Det (2,4) = $d_{22}d_{44} – x_{24}x_{42}$
Det (2,5) = $d_{22}d_{55} – x_{25}x_{52,}$ Det (3,4) = $d_{33}d_{44} – x_{34}x_{43,}$ Det (3,5) = $d_{33}d_{55} – x_{35}x_{53}$
Det (4,5) = $d_{44}d_{55} – x_{45}x_{54.}$

Table 10. Determinants of 2×2 sub-matrices

Principal submatrix	**Principal minor**
A (1,2)	Det A (1,2) = $(d_{11}d_{22}) – (a_{12}a_{21}) \geq 0$
A (1,3)	Det A (1,3) = $d_{11}d_{33} \geq 0$.
A (1,4)	Det A (1,4) = $d_{11}d_{4\,4} \geq 0$.
A (1,5)	Det A (1,5) = $d_{11}d_{55} \geq 0$.
A (2,3)	Det A (2,3) =$d_{22}d_{33} \geq 0$.
A (2,4)	Det A (2,4) = $d_{22}d_{44} \geq 0$.
A (2,5)	Det A (2,5) = $d_{22}d_{55} \geq 0$.
A (3,4)	Det A (3,4) = $d_{33}d_{44} \geq 0$.
A (3,5)	Det A (3,5) = $d_{33}d_{55} \geq 0$.
A (4,5)	Det A (4,5) = $d_{44}d_{55} \geq 0$.

By definition of completion, determinants of 2 × 2 submatrices are obtained. i.e. Det (1,2) = $d_{11}d_{22}$ - $a_{12}a_{21} \geq 0$. Det (1,3) = $d_{11}d_{33}$ - $a_{13}x_{31} \geq 0$. Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij} =1 then; Det (1,2) = 1-1 ≥ 0, det A (1,3) = 1– 0 = 1 > 0. Therefore, all the determinants of 2 × 2 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Table 11. Determinants of 3×3 sub-matrices

Principal submatrix	**Principal minor**
A (1,2,3)	Det A (1,2,3) = d_{33} $(d_{11}d_{22} – a_{12}a_{21}) \geq 0$.
A (1,2,4)	Det A (1,2,4) = $d_{44}(d_{11}d_{22} – a_{12}a_{21}) \geq 0$.
A (1,2,5)	Det A (1,2,5) = $d_{55}(d_{11}d_{22} – a_{12}a_{21}) \geq 0$.
A (1,3,4)	Det A (1,3,4) = $d_{11}d_{33}d_{44} \geq 0$.
A (1,3,5)	Det A (1,3,5) = $d_{11}d_{33}d_{55} \geq 0$.
A (1,4,5)	Det A (1,4,5) = $d_{11}d_{44}d_{55} \geq 0$.
A (2,3,4)	Det A = (2,3,4) $d_{22}d_{33}d_{44} \geq 0$.
A (2,3,5)	Det A (2,3,5) =$d_{22}d_{33}d_{55} \geq 0$.
A (2,4,5)	Det A (2,4,5) = $d_{22}d_{44}d_{55} \geq 0$.
A (3,4,5)	Det A (3,4,5) = $d_{33}d_{44}d_{55} \geq 0$.

Determinants of 3 × 3 submatrices are obtained. i.e., Det (3,4,5) = d_{11} $(d_{22}d_{33} – x_{23}\,x_{32})$ - a_{12} $(a_{21}\,d_{33} – x_{23}\,x_{31})$ + $a_{13}\,(a_{21}\,x_{32} – d_{22}\,x_{31}) \geq 0$. Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij} =1 then; det A (3,4,5) = 1– 1 + 0= 1 > 0. Therefore, all the determinants of 3 × 3 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Table 12. Determinants of 4×4 sub-matrices

Principal sub-matrix	**Principal minor**
A (1,2,3,4)	Det A (1,2,3,4) = $d_{33}d_{44}$ $(d_{11}d_{22} – a_{12}a_{21}) \geq 0$.
A (1,2,3,5)	Det A (1,2,3,5) = $d_{33}d_{55}(d_{11}d_{22} – a_{12}a_{21}) \geq 0$.
A (1,2,4,5)	Det A (1,2,4,5) = $d_{33}d_{55}(d_{11}d_{22} – a_{12}a_{21}) \geq 0$.
A (1,3,4,5)	Det A (1,3,4,5) = $d_{11}d_{33}d_{44}d_{55} \geq 0$.
A (2,3,4,5)	Det A (2,3,4,5) = $d_{22}d_{33}d_{44}d_{55} \geq 0$.

Det (A) = $d_{11}d_{22}d_{33}d_{44}d_{55} – (a_{12}a_{21}d_{33}d_{44}d_{55}) \geq 0$. Therefore, Det (A) =1 – 1 = 0.

Determinants of 4 × 4 submatrices are obtained. i.e., Det (1,2,3,4) = $d_{11}[d_{22}\,(d_{33}\,d_{44} – x_{34}\,x_{43})$ - $x_{23}\,(d_{44}\,x_{32} – x_{34}\,x_{42})$ + $x_{24}\,(x_{32}\,x_{43} – d_{33}\,x_{42})]$

$– a_{12}\,[a_{21}\,(d_{33}\,d_{44} – x_{34}\,x_{43})$ - $x_{23}\,(d_{44}\,x_{31} – x_{34}\,x_{41}) + x_{24}\,(x_{31}\,x_{43} – d_{33}\,x_{41})]$
$+ a_{13}\,[a_{21}\,(x_{32}\,d_{44} – x_{34}\,x_{42}) – d_{22}\,(d_{44}\,x_{31} – x_{34}\,x_{41}) + x_{24}\,(x_{31}\,x_{42} – x_{32}\,x_{41})]$

$- x_{14} [a_{21} (x_{32} x_{43} - d_{33} x_{42}) - d_{22}(x_{43} x_{31} - d_{33} x_{41}) + x_{23} (x_{31} x_{42} - x_{32} x_{41})] \geq 0.$

Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij} =1 then; det A (1,2,3,4) = 1 – 1 + 0 – 0 = 0. Therefore, all the determinants of 4 × 4 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Digraph D of order 5 and 4 arcs; consider a digraph D = {(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (3,5), (4,4), (5,5)} with 5 vertices and 4 arcs given by:

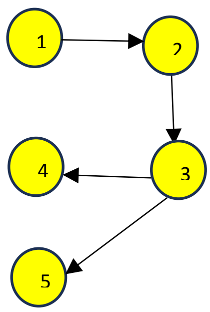

Fig. 5. Digraph D of order 5 and 4 arcs

The Partial matrix that specifies the digraph D is $\boldsymbol{A} = \begin{pmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & a_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$.By definition of partial Wss P_o- matrix $d_1 \geq 0$, $d_2 \geq 0$, $d_3 \geq 0$, $d_4 \geq 0$, $d_5 \geq 0$.

For d_{ii} = 1, a_{ij} = 1, then the partial matrix becomes A = $\begin{pmatrix} 1 & 1 & x_{13} & x_{14} & x_{15} \\ x_{21} & 1 & 1 & x_{24} & x_{25} \\ x_{31} & x_{32} & 1 & 1 & 1 \\ x_{41} & x_{42} & x_{43} & 1 & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix}$Next, we compute the principal minors.

Det (1,2) = $d_{11} d_{22} - a_{12} x_{21}$ Setting unspecified entries to zero, d_{ii} =1, a_{ij} =1, then we have; 1– 0 = 0. Similarly; Det (1,3) = $d_{11} d_{33} - x_{13} x_{31}$, Det (1,4) = $d_{11} d_{44} - x_{14} x_{41}$, Det (1,5) = $d_{11} d_{55} - x_{15} x_{51}$, Det (2,3) = $d_{22} d_{33} - a_{23} x_{32}$, Det (2,4) = $d_{22} d_{44} - x_{24} x_{42}$, Det (2,5) = $d_{22} d_{55} - x_{25} x_{52}$, Det (3,4) = $d_{33} d_{44} - a_{34} x_{43}$, Det (3,5) = $d_{33} d_{55} - a_{35} x_{53}$
Det (4 5) = $d_{44} d_{55} - x_{45} x_{54}$.

Table 13. Determinants of 2×2 sub-matrices

Principal submatrix	Principal minor
A (1,2)	Det A (1,2) = $d_{11}d_{22} \geq 0$.
A (1,3)	Det A (1,3) = $d_{11}d_{33} \geq 0$.
A (1,4)	Det A (1,4) = $d_{11}d_{44} \geq 0$.
A (1,5)	Det A (1,5) = $d_{11}d_{55} \geq 0$.
A (2,3)	Det A (2,3) = $d_{22}d_{33} \geq 0$.
A (2,4)	Det A (2,4) = $d_{22}d_{44} \geq 0$.
A (2,5)	Det A (2,5) = $d_{22}d_{55} \geq 0$.
A (3,4)	Det A (3,4) = $d_{33}d_{44} \geq 0$.
A (3,5)	Det A (3,5) = $d_{33}d_{55} \geq 0$.
A (4,5)	Det A (4,5) = $d_{44}d_{55} \geq 0$.

By definition of completion, determinants of 2 × 2 submatrices are obtained. i.e. Det (1,2) = $d_{11}d_{22} - a_{12}x_{21} \geq 0$. Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij} =1 then; det A (1,2) = 1– 0 = 1 > 0. Therefore, all the determinants of 2 × 2 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Table 14. Determinants of 3×3 sub-matrices

Principal submatrix	Principal minor
A (1,2,3)	Det A (1,2,3) = $d_{11}d_{22}d_{33} \geq 0$.
A (1,2,4)	Det A (1,2,4) = $d_{11}d_{22}d_{44} \geq 0$.
A (1,2,5)	Det A (1,2,5) = $d_{11}d_{22}d_{55} \geq 0$.
A (1,3,4)	Det A (1,3,4) = $d_{11}d_{33}d_{44} \geq 0$.
A (1,3,5)	Det A (1,3,5) = $d_{11}d_{33}d_{55} \geq 0$.
A (1,4,5)	Det A (1,4,5) = $d_{11}d_{44}d_{55} \geq 0$.
A (2,3,4)	Det A = (2,3,4) $d_{22}d_{33}d_{44} \geq 0$.
A (2,3,5)	Det A (2,3,5) =$d_{22}d_{33}d_{55} \geq 0$.
A (2,4,5)	Det A (2,4,5) = $d_{22}d_{44}d_{55} \geq 0$.
A (3,4,5)	Det A (3,4,5) = $d_{33}d_{44}d_{55} \geq 0$.

Determinants of 3 × 3 submatrices are obtained. i.e., Det (1,2,3) = d_{11} ($d_{22}d_{33} - a_{23}\, x_{32}$) - a_{12} ($a_{21}\, d_{33} - a_{23}\, x_{31}$) + x_{13} ($x_{21}\, x_{32} - d_{22}\, x_{31}$) ≥ 0. Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij} =1 then; det A (1,2,3) = 1– 1 + 0= 1 > 0. Therefore, all the determinants of 3 × 3 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Table 15. Determinants of 4×4 sub-matrices

Principal sub-matrix	Principal minor
A (1,2,3,4)	Det A (1,2,3,4) = $d_{11}d_{22}d_{33}d_{44} \geq 0$.
A (1,2,3,5)	Det A (1,2,3,5) = $d_{11}d_{22}d_{33}d_{55} \geq 0$.
A (1,2,4,5)	Det A (1,2,4,5) = $d_{11}d_{22}d_{44}d_{55} \geq 0$.
A (1,3,4,5)	Det A (1,3,4,5) = $d_{11}d_{33}d_{44}d_{55} \geq 0$.
A (2,3,4,5)	Det A (2,3,4,5) = $d_{22}d_{33}d_{44}d_{55} \geq 0$.

Det (A) = $d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$. Therefore Det (A) = 1 > 0.

Determinants of 4 × 4 submatrices are obtained. i.e., Det (1,2,3,4) = d_{11}[d_{22} ($d_{33}\, d_{44} - a_{34}\, x_{43}$) - a_{23} ($d_{44}\, x_{32} - a_{34}\, x_{42}$) + x_{24} ($x_{32}\, x_{43} - d_{33}\, x_{42}$)]

$-\, a_{12}\, [x_{21}\, (d_{33}\, d_{44} - a_{34}\, x_{43}) - a_{23}\, (d_{44}\, x_{31} - a_{34}\, x_{41}) + x_{24}\, (x_{31}\, x_{43} - d_{33}\, x_{41})]$
$+\, x_{13}\, [x_{21}\, (x_{32}\, d_{44} - a_{34}\, x_{42}) - d_{22}\, (d_{44}\, x_{31} - a_{34}\, x_{41}) + x_{24}\, (x_{31}\, x_{42} - x_{32}\, x_{41})]$
$-\, x_{14}\, [x_{21}\, (x_{32}\, x_{43} - d_{33}\, x_{42}) - d_{22}(x_{43}\, x_{31} - d_{33}\, x_{41}) + a_{23}\, (x_{31}\, x_{42} - x_{32}\, x_{41})] \geq 0$. Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij} =1 then; det A (1,2,3,4) = 1 – 0 + 0 – 0 = 1 > 0. Therefore, all the determinants of 4 × 4 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Digraph D of order 5 and 5 arcs: consider a digraph D = {(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,4), (4,5), (5,1), (5,5)} with 5 vertices and 5 arcs given by:

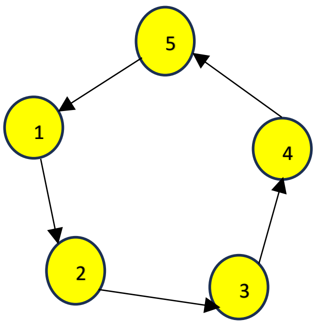

Fig. 6. Digraph D of order 5 and 5 arcs

The partial matrix that specifies the digraph D is $A = \begin{pmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ a_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$.

By definition of partial Wss P_o- matrix $d_1 \geq 0$, $d_2 \geq 0$, $d_3 \geq 0$, $d_4 \geq 0$, $d_5 \geq 0$. By definition of partial Wss P_o- matrix $d_1 \geq 0$, $d_2 \geq 0$, $d_3 \geq 0$, $d_4 \geq 0$, $d_5 \geq 0$.

For $d_{ii} = 1$, $a_{ij} = 1$, then the partial matrix becomes A = $\begin{pmatrix} 1 & 1 & x_{13} & x_{14} & x_{15} \\ x_{21} & 1 & 1 & x_{24} & x_{25} \\ x_{31} & x_{32} & 1 & 1 & x_{35} \\ x_{41} & x_{42} & x_{43} & 1 & 1 \\ 1 & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix}$. Next, we compute the principal minors.

Det (1,2) = $d_{11}\, d_{22}$ - $a_{12}\, x_{21}$. Det (1,2) = $d_{11}\, d_{22}$ - $a_{12}\, x_{21}$ Setting unspecified entries to zero, d_{ii} =1, a_{ij} =1, then we have; 1– 0 = 1 >0. similarly; Det (1,3) = $d_{11}\, d_{33}$ - $x_{13}\, x_{31}$, Det (1,4) = $d_{11}\, d_{44}$ - $x_{14}\, x_{41}$
Det (1,5) = $d_{11}\, d_{55}$ - $x_{15}\, a_{51}$, Det (2,3) = $d_{22}\, d_{33} - a_{23}\, x_{32}$, Det (2,4) = $d_{22}\, d_{44} - x_{24}\, x_{42}$
Det (2,5) = $d_{22}\, d_{55} - x_{25}\, x_{52}$, Det (3,4) = $d_{33}\, d_{44} - a_{34}\, x_{43}$, Det (3,5) = $d_{33}\, d_{55} - x_{35}\, x_{53}$
Det (4 5) = $d_{44}\, d_{55} - a_{45}\, x_{54}$.

Table 16. Determinants of 2×2 sub-matrices

Principal submatrix	Principal minor
A (1,2)	Det A (1,2) = $d_{11}d_{22} \geq 0$.
A (1,3)	Det A (1,3) = $d_{11}d_{33} \geq 0$.
A (1,4)	Det A (1,4) = $d_{11}d_{4\,4} \geq 0$.
A (1,5)	Det A (1,5) = $d_{11}d_{55} \geq 0$.
A (2,3)	Det A (2,3) =$d_{22}d_{33} \geq 0$.
A (2,4)	Det A (2,4) = $d_{22}d_{44} \geq 0$.
A (2,5)	Det A (2,5) = $d_{22}d_{55} \geq 0$.
A (3,4)	Det A (3,4) = $d_{33}d_{44} \geq 0$.
A (3,5)	Det A (3,5) = $d_{33}d_{55} \geq 0$.
A (4,5)	Det A (4,5) = $d_{44}d_{55} \geq 0$.

Determinants of 2 × 2 submatrices are obtained. i.e. Det (1,2) = $d_{11}d_{22} - a_{12}x_{21} \geq 0$. Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij} =1 then; det A (1,2) = 1– 0 = 1 > 0. Therefore, all the determinants of 2 × 2 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Table 17. Determinants of 3×3 sub-matrices

Principal submatrix	Principal minor
A (1,2,3)	Det A (1,2,3) = $d_{11}d_{22}d_{33} \geq 0$.
A (1,2,4)	Det A (1,2,4) = $d_{11}d_{22}d_{44} \geq 0$.
A (1,2,5)	Det A (1,2,5) = $d_{11}d_{22}d_{55} \geq 0$.
A (1,3,4)	Det A (1,3,4) = $d_{11}d_{33}d_{44} \geq 0$.
A (1,3,5)	Det A (1,3,5) = $d_{11}d_{33}d_{55} \geq 0$.
A (1,4,5)	Det A (1,4,5) = $d_{11}d_{44}d_{55} \geq 0$.
A (2,3,4)	Det A = (2,3,4) $d_{22}d_{33}d_{44} \geq 0$.
A (2,3,5)	Det A (2,3,5) =$d_{22}d_{33}d_{55} \geq 0$.
A (2,4,5)	Det A (2,4,5) = $d_{22}d_{44}d_{55} \geq 0$.
A (3,4,5)	Det A (3,4,5) = $d_{33}d_{44}d_{55} \geq 0$.

Determinants of 3 × 3 submatrices are obtained. i.e., Det (1,2,3) = $d_{11}\,(d_{22}d_{33} - a_{23}\,x_{32})$ - $a_{12}\,(x_{21}\,d_{33} - a_{23}\,x_{31})$ + $x_{13}\,(x_{21}\,x_{32} - d_{22}\,x_{31}) \geq 0$. Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij} =1 then; det A (1,2,3) = 1– 0 + 0 = 1 > 0. Therefore, all the determinants of 3 × 3 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

Table 18. Determinants of 4×4 sub-matrices

Principal sub-matrix	Principal minor
A (1,2,3,4)	Det A (1,2,3,4) = $d_{11}d_{22}d_{33}d_{44} \geq 0$.
A (1,2,3,5)	Det A (1,2,3,5) = $d_{11}d_{22}d_{33}d_{55} \geq 0$.
A (1,2,4,5)	Det A (1,2,4,5) = $d_{11}d_{22}d_{44}d_{55} \geq 0$.
A (1,3,4,5)	Det A (1,3,4,5) = $d_{11}d_{33}d_{44}d_{55} \geq 0$.
A (2,3,4,5)	Det A (2,3,4,5) = $d_{22}d_{33}d_{44}d_{55} \geq 0$.

Det (A) = $(d_{11}d_{22}d_{33}d_{44}d_{55}) - (a_{12}a_{23}a_{34}a_{45}a_{51}) \geq 0$. Therefore Det (A) = 1 – 1 = 0. By definition of completion, determinants of 4 × 4 submatrices are obtained. i.e., Det (1,2,3,4) = $d_{11}[d_{22}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(d_{44}x_{32} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})]$

$$- a_{12}[x_{21}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(d_{44}x_{31} - a_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})]$$
$$+ x_{13}[x_{21}(x_{32}d_{44} - a_{34}x_{42}) - d_{22}(d_{44}x_{31} - a_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})]$$
$$- x_{14}[x_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{43}x_{31} - d_{33}x_{41}) + a_{23}(x_{31}x_{42} - x_{32}x_{41})] \geq 0.$$

Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and a_{ij} =1 then; det A (1,2,3,4) = 1 – 0 + 0 – 0 = 1 > 0. Therefore, all the determinants of 4 × 4 submatrices are non-negative then the partial matrix can be completed into W**ss** P_o-matrix. Hence it has zero completion into a W**ss** P_o-matrix.

All the digraphs mentioned above are not isomorphic to each other.

4. Conclusion

Upon comprehensive examination and analysis of various digraph patterns, the research arrived at significant conclusions. We used the zero-completion method where unspecified entries in partial Wss P_o- matrices are set to zero to evaluate principal minors. We concluded that a null graph of order 5 has Wss P_o- completion. Additionally, we demonstrated that digraphs of order 5 with one to five arcs whether cyclic, acyclic, or with cliques, can be completed into a Wss P_o- matrix. Therefore, we found that all partial Wss P_o- matrices have completion into a Wss P_o- matrix. Conversely, no partial Wss P_o- matrix formed by digraphs of order 5 with up to 5 arcs lacks the ability to be completed into a Wss P_o- matrix. Insights gained from these studies could be applied to fill gaps in data from retail surveys, aiding in the prediction of market trends.

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Competing Interests

Authors have declared that no competing interests exist.

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