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Modified Adomain Decomposition Method for the Generalized Fifth Order KdV Equations

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ABSTRACT

New modified Adomian decomposition method is proposed for the solution of the generalized fifth-order Korteweg-de Vries (GFKdV) equation. The numerical solutions are compared with the standard Adomian decomposition method and the exact solutions. The results are demonstrated which confirm the efficiency and applicability of the method.

Keywords: Modified Adomian Decomposition Method; Fifth Order KdV Equation; Solitary-Wave Solution

1. Introduction

The fifth-order (or generalized) KdV is the essential a model foe several physical phenomena including shallow-water waves near critical value of surface tension and waves in nonlinear LC circuit with mutual inductance between neighbouring inductors [1]. Although no general solution is known, the exact solution of the fifth order KdV equation has been found for the special case of solitary waves in [2]. In general, the fifth order KdV need to be solved numerically. Commonly used numerical methods to approximate the solutions of (gfKdV) include finite difference methods, collection methods and Galerkin methods. Kawahara [3] investigated the steady solutions of this equation on the basis of numerical calculations. Boyd [4] and Haupt and Boyd [5] reduced this equation to an ordinary differential equation and then a variety of analyticl and numerical methods are developed. The numerical methodes are based on Newton-Kantorovich pseudo-spectral method and Newton-Kantorovich Galerkin method. In [6] K. Djidjeli *et al*. proposed finite deference schemes based on a predictor - corrector algorithm and a linearized implicit method for the third and fifth order KdV equations.

However, some of these methods are not easy to use and sometimes require tedious work and calculation [7, 8]. In recent years, Adomian decomposition methods (ADM) [9] have emerged as a powerful tool for a wide class of nonlinear equation [10]. G. Adomian in [11] applied his method to the 5th order KdV equation. In [12, 13], D. Kaya calculated the explicit and numerical solutions of some fifth-order KdV equation by decomposition method and Kaya and El-Sayed in [14] proved the convergence of (ADM) applied to (gfKdV) equation.

A comparative study between (ADM) and Crank Nicholas method presented in [15]. (ADM) has led to several modifications on the method made by various researches in order to improve the accuracy or expansion of the application of the original method. Wazwaz [16] presented a reliable modification of the Adomian decomposition method.

In 2001, Wazwaz presented another type of modification [17] to the (ADM). Wazwaz modifications arises in the initial definition of the operator when applying the (ADM) to the nonlinear equation.

To the best of our knowledge, no attempt is made regarding the solution of generalized fifth order KdV equations by using modified decomposition method. So our main aim in this paper is used the modification ADM to solve the five particular class of the (gfKdV) equations. We generalized an appropriate Adomian's polynomials for (gfKdV) equation will be handle more easily, quickly, and elegantly by implementing the new modified (ADM) rather than traditional methods for the exact solution of which is to be obtained subject to initial condition.

2. Fifth-Order KdV Equations

The well-known fifth-order KdV (fKdV) equations can be shown in the form

$$u_t + au^2u_x + bu_xu_{xx} + cuu_{xxx} + du_{xxxxx} = 0 \qquad (1)$$

where a, b, c and d are nonzeros and real parameters, and $u = u(x,t)$ is a sufficiently smooth function. The (fKdV) is an important mathematical model with wide applications in quantum mechanics and nonlinear optics.

Typical examples widely used in various fields such as solid state physics, plasma physics, fluid physics and

quantum field theory. A variety of the (fKdV) equations can be developed by changing the real values of the parameters a, b and c [18]. the derivation of these fifth-order forms are derived specific bilinear forms of the so-called Hirota's D-operators. However, five will known forms of the (fKdV) that are of particular interest in the literature. These forms are:

1) The Sawada-Kotera (SK) equation is given by [19]

$$u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx} = 0 \quad (2)$$

2) The Caudrey-Dodd-Gibbon (CDG) equation is given by [20]

$$u_t + 180u^2u_x + 30u_xu_{xx} + 30uu_{xxx} + u_{xxxxx} = 0 \quad (3)$$

3) The Lax equations [21]

$$u_t + 30u^2u_x + 30u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0 \quad (4)$$

4) The Kaup-Kuperschmidt (KK) equation [22,23]

$$u_t + 20u^2u_x + 25u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0 \quad (5)$$

5) The Ito equation [24]

$$u_t + 2u^2u_x + 6u_xu_{xx} + 3uu_{xxx} + u_{xxxxx} = 0 \quad (6)$$

3. The Method of Soultion

3.1. Adomian Decomposition Method

In this section, we give outline and implement Adomian decomposition method for nonlinear equations to obtain analytic and approximate solutions which are obtained in a rapidly convergent series with elegantly computable components by this method. The Adomian approximation series converge quickly. In general, convergence regions of the series are small. Now we outline of the method here in order to obtain the solutions using (ADM), consider the fifth-order KdV Equation (1) in an operator form

$$L_t(u) + aK_u + bM_u + cN_u + dL_xu = 0, \quad (7)$$

where the notations $K_u = u^2u_x$, $M_u = u_xu_{xx}$, and $N_u = uu_{xxx}$. By symbolizing the nonlinear term, respectively. The notation $L_t = \frac{\delta}{\delta t}$ and $L_x = \frac{\delta^5}{\delta x^5}$ symbolize the linear differential operators. Assuming the inverse of operator L_t^{-1} exists and it can conveniently be taken as the definite integral with respect to t from 0 to t, *i.e.*,

$$L_t^{-1} = \int_0^t (\cdot)\,\mathrm{d}t \quad (8)$$

Thus, applying the inverse operator L_t^{-1} to (7) yields;

$$\begin{aligned} &L_t^{-1}L_t(u) = -aL_t^{-1}(K_u) - bL_t^{-1}(M_u) \\ &-cL_t^{-1}(N_u) - dL_t^{-1}(L_xu) = 0, \end{aligned} \quad (9)$$

Therefore, it follows that

$$\begin{aligned} &u(x,t) - u(x,0) \\ &= -aL_t^{-1}(K_u) - bL_t^{-1}(M_u) - cL_t^{-1}(N_u) - dL_t^{-1}(L_xu), \end{aligned} \quad (10)$$

Since initial value is known and we decompose the unknown function $u(x,t)$ as a sum of components defined by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (11)$$

with u_0 identified as $u(x,0)$.

The nonlinear terms K_u, M_u and N_u can be decomposed into infinite series of polynomial given by

$$K_u = u^2u_x = \sum_{n=0}^{\infty} A_n, \quad (12)$$

$$M_u = u_xu_{xx} = \sum_{n=0}^{\infty} B_n, \quad (13)$$

$$N_u = uu_{xxxx} = \sum_{n=0}^{\infty} C_n, \quad (14)$$

A_n, B_n, and C_n are the so-called Adomian polynomials of $u_0, u_1, \cdots, u_n$. defined by

$$P_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[\Phi\left(\sum_{k=1}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0}, \quad n = 0,1,2,\cdots. \quad (15)$$

Substituting (11-14) into (10) gives rise to

$$\begin{aligned} &u_{n+1} = -aL_t^{-1}A_n - bL_t^{-1}B_n - cL_t^{-1}C_n - dL_t^{-1}(L_xu_n), \\ &n \ge 1. \end{aligned} \quad (16)$$

The solution $u(x,t)$ must satisfy the requirements imposed by the initial conditions. Based on the (ADM), we constructed the solution $u(x,t)$ as

$$\begin{aligned} &\lim_{n\to\infty}\varphi_n = u(x,t), \\ &\text{where } \varphi_n(x,t) = \sum_{k=0}^{n} u_k(x,t), \ n \ge 0. \end{aligned} \quad (17)$$

The decomposition method provides a reliable technique that requires less work if we compared with the traditional techniques.

3.2. New Modified Adomian Decomposition Method

In the new modification by Wazwaz [17], we can replace the process of identified u_0 as $u(x,0)$ by dividing $u(x,0)$ by a series of infinite components. We therefore suggest that

$$u_0(x,t) = \sum_{n=0}^{\infty} u_0^n(x). \quad (18)$$

A new recursive relationship expressed in the form

$$u_0 = u_0^0$$

$$u_{n+1} = u_0^{n+1} - aL_t^{-1}A_n - bL_t^{-1}B_n - cL_t^{-1}C_n - dL_t^{-1}\left(L_x u_n\right), \quad n \geq 1. \tag{19}$$

We can observe that algorithm (19) reduces the number of terms involved in each component, and hence the size of calculations is minimized compared to the standard (ADM) only. Moreover this reduction of terms in each component facilitates the construction of Adomian polynomials for nonlinear operators. the new modification overcomes the difficulty of decomposing $u(x,0)$ and introduces an efficient algorithm that improves the performance of the standard (ADM).

4. Numerical Experiments

In this section, we consider some (gfKdV) equations for numerical comparisons based on the new modifications of (ADM). In this paper, we illustrate how the approximate solutions of the (gfKdV) equations are close to exact solutions.

4.1. Example (1): (Sawada-Kotera Equation)

we consider the (S-K) Equation [25], with initial condition is given by

$$u_0(x,t) = 2k^2 \operatorname{sech}^2\left(k\left(x - x_0\right)\right). \tag{20}$$

and the exact solution

$$u(x,t) = 2k^2 \operatorname{sech}^2\left(k\left(x - 16k^4 t - x_0\right)\right). \tag{21}$$

Table 1 shows the difference of the analytical solution and numerical solution of the absolute errors only for 5 iterative.

4.2. Example (2): (Caudrcy-Dodd-Gbbon (C-D-G) Equation)

we consider the (C-D-G) equation, with initial condition

$$u(x,0) = \frac{k^2 e^{kx}}{\left(1 + e^{kx}\right)^2}. \tag{22}$$

and the exact solution

$$u(x,t) = \frac{k^2 e^{\left(k\left(x - k^4 t\right)\right)}}{\left[1 + e^{\left(k\left(x - k^4 t\right)\right)}\right]^2}. \tag{23}$$

Table 2 shows the numerical results for example (2) for $k = 0.01$.

4.3. Example (3): (The Lax Equation)

we consider Lax's fifth order KdV equation with the initial condition:

$$u(x,0) = 2k^2\left(2 - 3\tanh^2\left(k\left(x - x_0\right)\right)\right). \tag{24}$$

and the exact solution

$$u(x,t) = 2k^2\left(2 - 3\tanh^2\left(k\left(x - 56k^4 t - x_0\right)\right)\right). \tag{25}$$

Table 3 shows the numerical results for example (3) for $k = 0.01$, $x_0 = 0.0$.

4.4. Example (4): (Kaup-Kuperschmidt (K-K) Equation)

We consider the (K-K) equation with the initial condition

$$u(x,0) = \frac{24k^2 e^{kx}\left[4e^{kx} + e^{2kx} + 16\right]}{\left[16e^{kx} + e^{2kx} + 16\right]^2}. \tag{26}$$

and the exact solution

$$u(x,t) = \frac{24k^2 e^{kx-t}\left[4e^{kx-k^5 t} + e^{2kx-k^5 t} + 16\right]}{\left[16e^{kx-k^5 t} + e^{2\left(kx-k^5 t\right)} + 16\right]^2}. \tag{27}$$

Table 4 shows the numerical results for example (4) for $k = 0.01$.

4.5. Example (5): (Ito Equation)

we consider the Ito equation with the initial condition

$$u(x,0) = 20k^2 - 30k^2 \operatorname{tanch}^2\left[kx\right]. \tag{28}$$

and the exact solution

$$u(x,t) = 20k^2 - 30k^2 \operatorname{tanch}^2\left[kx - 96k^4 + x_0\right]. \tag{29}$$

Table 5 shows the numerical results for example (5) for $k = 0.01$ and $x_0 = 0.0$.

5. Conclusions and Remarks

In this work, we proposed new modification of Adomian decomposition method. We solved the five well known forms of the (fKdV) equation with initial conditions. The method has been shown to computationally efficient in these examples that are important to researchers in applied sciences. The obtained results in examples indicted that the new modification of (ADM) was feasible and effective. The method overcomes the difficulties arising in the modified decomposition method established in [16].

The results show that the presented method is powerful mathematical tool for finding good approximate solu-

Table 1. Absolute error between the exact solution and approximation solution for $k = 0.01$ and $x_0 = 0.0$.

t/x	0.2	0.4	0.6	0.8	1.0	5.0
2	1.54499×10^{-18}	4.52654×10^{-18}	1.25496×10^{-17}	2.81893×10^{-17}	5.40204×10^{-17}	6.6435×10^{-15}
4	5.36680×10^{-18}	1.12757×10^{-17}	2.7349×10^{-17}	5.87095×10^{-17}	1.10345×10^{-16}	1.32874×10^{-14}
6	1.13841×10^{-17}	2.02746×10^{-17}	4.44252×10^{-17}	9.14253×10^{-17}	1.68892×10^{-16}	1.99336×10^{-14}
8	1.97054×10^{-17}	3.15232×10^{-17}	6.37511×10^{-17}	1.26364×10^{-16}	2.29688×10^{-16}	2.6582×10^{-14}
10	3.02492×10^{-17}	4.50486×10^{-17}	8.52725×10^{-17}	1.63633×10^{-16}	2.92707×10^{-16}	3.32327×10^{-14}

Table 2. Absolute error between the exact solution and approximation solution for $k = 0.01$ and $x_0 = 0.0$.

t/x	0.2	0.4	0.6	0.8	1.0	5.0
2	1.96921×10^{-21}	3.11957×10^{-21}	3.63542×10^{-21}	1.7893×10^{-20}	2.62474×10^{-20}	2.48025×10^{-18}
4	3.87141×10^{-21}	2.70138×10^{-21}	9.52285×10^{-21}	2.01336×10^{-20}	5.33583×10^{-20}	5.70681×10^{-18}
6	7.39739×10^{-21}	5.50257×10^{-21}	2.20177×10^{-20}	3.91461×10^{-20}	7.69124×10^{-20}	8.93996×10^{-18}
8	1.89429×10^{-21}	4.74685×10^{-21}	2.75676×10^{-20}	5.79897×10^{-20}	9.69092×10^{-20}	1.21797×10^{-17}
10	6.82819×10^{-21}	7.21049×10^{-21}	2.61724×10^{-20}	5.63358×10^{-20}	1.20126×10^{-19}	1.54092×10^{-17}

Table 3. The exact and approximation solution of Lax equation for $k = 0.01$.

t/x	0.2	0.4	0.6	0.8	1.0	5.0
2	5.76197×10^{-14}	1.15211×10^{-13}	1.72788×10^{-13}	2.30342×10^{-13}	2.87867×10^{-13}	1.42104×10^{-12}
4	1.15281×10^{-13}	2.30464×10^{-13}	3.45618×10^{-13}	4.60727×10^{-13}	5.75777×10^{-13}	2.84213×10^{-12}
6	1.72985×10^{-13}	3.45759×10^{-13}	5.18489×10^{-13}	6.91153×10^{-13}	8.63729×10^{-13}	4.26326×10^{-12}
8	2.3073×10^{-13}	4.61096×10^{-13}	6.91403×10^{-13}	9.21621×10^{-13}	1.15172×10^{-12}	5.68444×10^{-12}
10	2.88518×10^{-13}	5.76475×10^{-13}	8.64358×10^{-13}	1.15213×10^{-12}	1.43976×10^{-12}	7.10566×10^{-12}

Table 4. Absolute error between the exact solution and approximation solution for $k = 0.01$ and $x_0 = 0.0$.

t/x	0.2	0.4	0.6	0.8	1.0	5.0
2	1.46896×10^{-16}	7.86155×10^{-17}	1.0355×10^{-17}	5.77728×10^{-17}	1.25763×10^{-16}	2.24215×10^{-15}
4	2.9378×10^{-16}	1.57193×10^{-16}	2.07239×10^{-17}	1.15638×10^{-16}	2.51752×10^{-16}	7.63093×10^{-16}
6	4.40678×10^{-16}	2.35791×10^{-16}	3.10725×10^{-17}	1.73475×10^{-16}	3.77733×10^{-16}	7.15916×10^{-16}
8	5.87582×10^{-16}	3.14383×10^{-16}	4.14144×10^{-17}	2.31306×10^{-16}	5.03728×10^{-16}	2.19495×10^{-15}
10	7.34466×10^{-16}	3.92994×10^{-16}	5.17698×10^{-17}	2.89164×10^{-16}	6.29737×10^{-16}	3.67398×10^{-15}

Table 5. Absolute error between the exact solution and approximation solution for $k = 0.01$ and $x_0 = 0.0$.

t/x	0.2	0.4	0.6	0.8	1.0	5.0
2	1.40995×10^{-16}	1.41060×10^{-16}	1.41178×10^{-16}	1.4137×10^{-16}	1.4168×10^{-16}	7.17504×10^{-16}
4	5.63981×10^{-16}	5.64238×10^{-16}	5.64685×10^{-16}	5.65365×10^{-16}	5.66365×10^{-16}	1.74347×10^{-15}
6	1.26896×10^{-15}	1.26953×10^{-15}	1.27052×10^{-15}	1.27199×10^{-15}	1.27405×10^{-15}	3.07794×10^{-15}
8	2.25592×10^{-15}	2.25695×10^{-15}	2.25868×10^{-15}	2.26123×10^{-15}	2.26475×10^{-15}	4.7209×10^{-15}
10	3.52488×10^{-15}	3.52648×10^{-15}	3.52918×10^{-15}	3.5331×10^{-15}	3.53845×10^{-15}	6.67236×10^{-15}

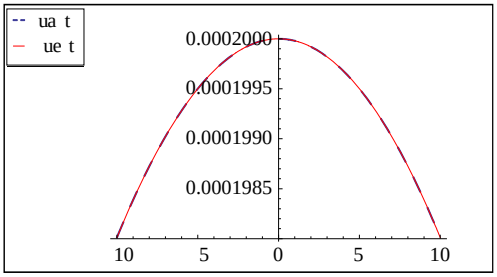

Figure 1. The exact and the approximation solution of s-k equation for *t* = 10 and *k* = 0.01.

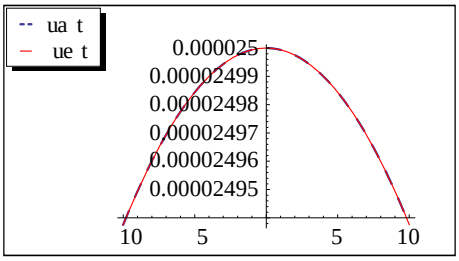

Figure 2. The exact and the approximation solution of C-D-G equation for *t* = 10 and *k* = 0.01.

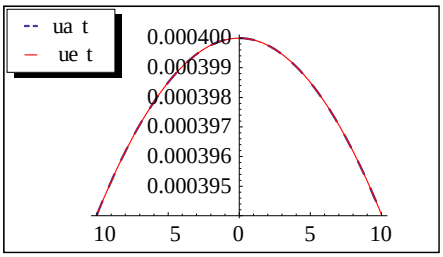

Figure 3. The exact and the approximation solution of Lax equation for *t* = 10 and *k* = 0.01.

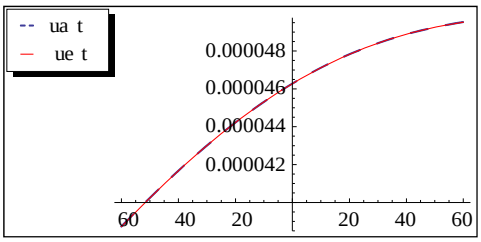

Figure 4. The exact and the approximation solution of K-K equation for *t* = 10 and *k* = 0.01.

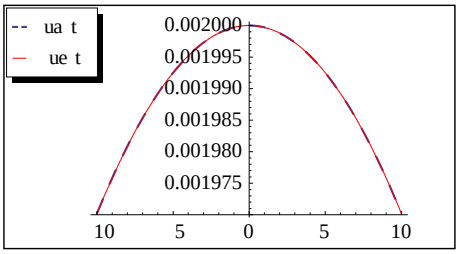

Figure 5. The exact and the approximation solution of Ito equation for *t* = 10 and *k* = 0.01.

tions of (gfKdV) equation with initial conditions and results are found to be in good agreement with the exact solution as shown from **Figures 1-5**. In addition, no linearization or perturbation is required by the method.

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