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Some Condition of Control on $\mathfrak{sl}(3, \mathbb{R})$

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper we present some original contributions to the problem of controllability of bilinear systems control whose dynamics is determined by elements that lie on the Lie algebra of special linear Lie group. Our study provides a sufficient condition for controllability of homogeneous bilinear systems, when the state variable dynamic modeling lies on the three-dimensional space. Such a condition is a contribution to the theory of geometric control.

Keywords: Biliinear systems; controllability; Lie algebra.

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1 Introduction

Controllability property of control systems is one of the most important problems in control theory and there is no general criterion. For example, if we considerer a linear system of the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$

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where $x(t) \in \mathbb{R}^n$, $t \in [t_0, T]$, $A \in M(n, \mathbb{R})$, $B \in M(n \times m, \mathbb{R})$ and u is piecewise constant function, there is a criterion known as *Kalman's criterion* to analyze the property of controllability. As is well know, many authors have discussed the controllability property for bilinear control systems when the state variable lies on the plane, see for instance [1], [2], [3]. However, when the dynamic is described on high dimension there are many unsolved problems, and in that sense the search conditions necessary and/or sufficient to characterize the property of controllability have attracted great attention during the last decades, [4] [5], [6], [7], [8], [9].

Jurdjevic-Kupka in [10], establish conditions for matrices A and B in such a way that the bilinear system is controllable on $SL(n, \mathbb{R})$. In the same way, Gauthier et Bornard in [11], as from [10], establish a necessary and sufficient condition for the bilinear control system.

In this paper, we have establish a characterization of controllability for the class of bilinear systems whose dynamics is determined by a special class of matrices that belong to the semisimple Lie algebra $\mathfrak{sl}(3, \mathbb{R})$

This paper is organized as it follows: in section 2 provides a review of theory of control on Lie Groups, matrices and graphs. In Section 3, it is presented the main result.

2 Preliminaries

The organization of the following definitions is presented so as this article is self- contained, for this reason we describe some basic concepts for the understanding of our work.

2.1 Matrices and graphs

Definition 2.1. Let $A = (a_{ij}) \in M(n, \mathbb{R})$ be, it defines the graph of A, denoted $\Gamma(A)$, as the set of n ordered points from 1 to n called vertices of the graph and oriented arcs formed by joining a vertex i to the vertex j, if $a_{ij} \neq 0$.

Example 2.1. *If* $A = \begin{pmatrix} 1 & 5 & -1 \\ 9 & -3 & 1 \\ 1 & -3 & 0 \end{pmatrix}$, *then* $\Gamma(A)$ *is:*

Definition 2.2. It is said that a graph $\Gamma(A)$ is strongly connected if for every pair (i, j) of vertices, there is exists an oriented path connecting i to j starting from i and ending in j.

Example 2.2. $\Gamma(A)$ *of Example 2.1 is strongly connected.*

Definition 2.3. A matrix $A \in M(n, \mathbb{R})$, is called permutation-reducible if there is exists a permutation matrix P such that:

$$P^{-1}AP = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

where $A_3 \in M(r, \mathbb{R})$ with $0 < r < n$.

Definition 2.4. A matrix $A \in M(n, \mathbb{R})$, is called permutation-irreducible if it is not permutation-reducible.

See more [12] and [13].

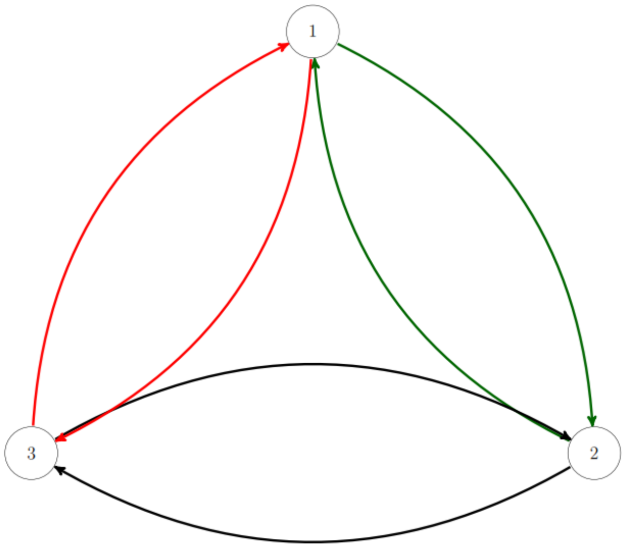

Fig. 1. $\Gamma(A)$ **Example 2.1**

Theorem 2.3. *$A \in M(n, \mathbb{R})$ is permutation-irreducible matrix if only if $\Gamma(A)$ is strongly connected.*

Proof. See [12] and [13]. □

2.2 Lie groups and Lie algebras

Definition 2.5. A Lie group G is smooth maniflod, abstrac group and group operations in G are smooth.

See more [14].

Example 2.4. *The special linear group consists of $n \times n$ unimodular matrices:*

$$SL\,(n, \mathbb{R}) = \{X \in M\,(n, \mathbb{R}) : det(X) = 1\}\,,$$

is a Lie group.

Definition 2.6. A Lie algebra $\mathfrak{g}$ is a vector space endowed with a binary operation

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g},$$

(Called Lie bracket) satisfying the following properties:

1. $[\cdot, \cdot]$ is bilinear.
2. $[x, x] = 0$, $\forall x \in \mathfrak{g}$
3. For all $x, y, z \in \mathfrak{g}$, $\quad [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$. (Jacobi identity)

Example 2.5. *$M\,(n, \mathbb{R})$ endowed of bracket defined by $[X, Y] = XY - YX$, is a Lie algebra.*

Definition 2.7. The tangent space to a Lie group G at the identity element is called the Lie algebra of the Lie group G. $\mathfrak{g} := T_{Id}G$.

Example 2.6. *The Lie algebra of the special linear group is denoted by*

$$\mathfrak{sl}(n,\mathbb{R}) := T_{Id}SL(n,\mathbb{R}) = \left\{\dot{X}(0) : X(t) \in SL(n,\mathbb{R}),\ \ X(0) = Id\right\}.$$

It is not difficult to see that $\mathfrak{sl}(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) : tr(A) = 0\}$.

Definition 2.8. A subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a Lie subalgebra if $[X,Y] \in \mathfrak{h}, \quad \forall X,Y \in \mathfrak{h}$.

In order to define exponential map, we remember that one-parameter subgroup of G is a function $\varphi : (\mathbb{R},+) \longrightarrow G$.

Definition 2.9. The function $exp : \mathfrak{g} \longrightarrow G$ is called exponential map of $\mathfrak{g}$ in the Lie group G and is define by $exp(X) := exp_X(1)$, where $t \mapsto exp_X(t)$ is a one-parameter subgroup.

If $G = GL(n,\mathbb{R})$, then $\mathfrak{g} = M(n,\mathbb{R})$ and exponential map is given by

$$exp(A) = e^A = Id + A + \frac{A^2}{2!} + \ldots + \frac{A^n}{n!} + \ldots,$$

with $n \in \mathbb{N}$ and $A \in M(n,\mathbb{R})$.

Definition 2.10. Let $\mathfrak{g}$ be a Lie algebra, its adjoint representation is the function

$$\begin{array}{cccc} \text{ad:} & \mathfrak{g} & \longrightarrow & \mathfrak{gl}(\mathfrak{g}) \\ & X & \longrightarrow & ad(X) \end{array}$$

where $ad(X)(Y) = [X,Y]$, for all $X,Y \in \mathfrak{g}$.

See more [15].

2.3 Controllability

As is known in the theory of differential equations they are studied equations depending on a parameter, such as

$$\dot{x} = f(x,u), \quad x(0) = x_0 \in M, \tag{2.1}$$

with the right side depending on the parameter u, which takes values in a set $U \subset \mathbb{R}^m$. The set U is called *the set of control parameters*, $x \in M$ represents the physical state of the system, f is a regular function, and $u(t) \in U \subset \mathbb{R}^m$ represents the income from the outside world, called *control*. When control u is fixed, the system equation $\dot{x} = f(x,u)$ a dynamic system is defined. Thus, the control system (2.1) it can be seen as a family of differential equations parameterized by the control as parameter.

We consider in this paper, a control system when f is linear in the control and variable states. That is, the system belongs to the class of bilinear systems. Namely, the control system bilinear unrestricted control, given by

$$\dot{x}(t) = (A + u(t)B)\,x(t), \tag{2.2}$$

where $A, B \in \mathfrak{sl}(n,\mathbb{R})$, control $u(t) \in \mathbb{R}$ and state variable $x \in \mathbb{R}^n \setminus \{0\}$. Equivalently,

$$\Gamma = A + uB \subset \mathfrak{sl}(n,\mathbb{R}), \quad u \in \mathbb{R}.$$

Definition 2.11. A trajectory of Γ on G is a continuous curve $\varphi(t)$ in G defined on an interval $[t_0,T] \subset \mathbb{R}$ so that there exists a partition $t_0 < t_1 < \ldots < t_N = T$ and $\Gamma_1, \ldots, \Gamma_N \in \Gamma$ such that the restriction of $\varphi(t)$ to each open interval (t_{i-1},t_i) is differentiable and

$$\dot{\varphi}(t) = \varphi(t)\Gamma_i \quad \text{for } t \in (t_{i-1},t_i),\ i = 1,\ldots,N.$$

Definition 2.12. For any $T > 0$ and any $g \in G$ the reachable set for time T of system $\Gamma \subset \mathfrak{g}$ from the point g is the set $\mathcal{A}_\Gamma(g, T)$, of all points that can be reached from g in exactly T units of time. More precisely,

$$\mathcal{A}_\Gamma(g, T) = \{\varphi(T) : \varphi(\cdot) \text{ a trajectory of } \Gamma, \varphi(0) = g\}.$$

The reachable set for time not greater than $T \geq 0$ is defined as

$$\mathcal{A}_\Gamma(g, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{A}_\Gamma(g, t).$$

The reachable (or attainable) set of a system Γ from a point $g \in G$ is the set,denoted $\mathcal{A}_\Gamma$, of all terminal points $\varphi(T)$, $T \geq 0$, of all trajectories of Γ starting at g, that is to say,

$$\mathcal{A}_\Gamma(g) = \bigcup_{T \geq 0} \mathcal{A}_\Gamma(g, \leq T).$$

In terms of the notion of attainable set we can introduce the property of controllability of a system as follows,

Definition 2.13. A system $\Gamma \subset \mathfrak{g}$ is called controllable if, given any pair of points g_0 and g_1 in a Lie group G, the point g_1 can be reached from g_0 along a trjectory of Γ for a nonnegative time:

$$g_1 \in \mathcal{A}_\Gamma(g_0), \quad \text{for any } g_0, g_1 \in G,$$

or in other words, if $\mathcal{A}_\Gamma(g) = G$ for any $g \in G$.

See more [16], [17] [18].

If $\Gamma \subset \mathfrak{g}$, we denoted by $Lie(\Gamma)$ the Lie algebra generated by Γ, i.e, the smallest Lie subalgebra of $\mathfrak{g}$ containing Γ.

Definition 2.14. A system $\Gamma \subset \mathfrak{g}$ is said to have full rank, or to satisfy the Lie algebra Rank Condition: LARC, if $Lie(\Gamma) = \mathfrak{g}$.

As indicated in [11], the full rank condition is equivalent to considering that the matrix A in the system (2.2), is permutation-irreducible.

The full rank condition is, in general, a necessary condition for the controllability of bilinear systems on Lie groups. If the Lie group G is compact for him controllability is equivalent to the LARC.

Theorem 2.7. *If Γ is controllable, then $Lie(\Gamma) = \mathfrak{g}$.*

Proof. See [17]. □

Definition 2.15. Let $\Gamma_1, \Gamma_2 \subset \mathfrak{g}$. The system Γ_1 is called equivalent to the system Γ_2, denoted $\Gamma_1 \sim \Gamma_2$ if

$$cl(\mathcal{A}_{\Gamma_1}) = cl(\mathcal{A}_{\Gamma_2}),$$

where, $cl(\cdot)$ denotes the closure topological

Definition 2.16. The saturate of a left-invariante system $\Gamma \subset \mathfrak{g}$ is the following systems:

$$Sat(\Gamma) = \bigcup \{\Gamma' \subset \mathfrak{g} : \Gamma' \sim \Gamma\}$$

Definition 2.17. The Lie saturate of a left-invariant system, denoted by $LS(\Gamma)$, is defined as follows:

$$LS(\Gamma) = Lie(\Gamma) \cap Sat(\Gamma)$$

Now, we will give a result that provides basic properties of Lie saturate of acontrol system.

Proposition 2.1. *(1) $LS(\Gamma)$ is closed convex positive cone in $\mathfrak{g}$, i.e,*

(1a) $LS(\Gamma)$ is topologically closed:

$$cl\left(LS\left(\Gamma\right)\right) = LS\left(\Gamma\right)$$

(1b) $LS(\Gamma)$ is convex:

$$A, B \in LS\left(\Gamma\right) \rightarrow \alpha A + (1-\alpha)\, B \in LS\left(\Gamma\right), \quad \forall \alpha \in [0,1]$$

(1c) $LS(\Gamma)$ is a positive cone:

$$A \in LS\left(\Gamma\right) \rightarrow \alpha A \in LS\left(\Gamma\right), \quad \forall \alpha \geq 0,$$

thus,

$$A, B \in LS\left(\Gamma\right) \rightarrow \alpha A + \beta B \in LS\left(\Gamma\right), \quad \forall \alpha, \beta \geq 0$$

(2) If $A, \pm B \in LS(\Gamma)$ for all $v \in \mathbb{R}$, then,

$$exp\left(v \cdot ad\left(B\right)\right) A \in LS\left(\Gamma\right)$$

(3) If $\pm A, \pm B \in LS(\Gamma)$, then $[A, B] \in LS(\Gamma)$.

(4) If $A \in LS(\Gamma)$ and if the one-parameter subgroup $\{exp\,(tX) : t \in \mathbb{R}\}$ is periodic (i.e, compact), then $-A \in LS(\Gamma)$.

Proof. See [16] □

3 Controllability Bilinear System on $\mathfrak{sl}(3, \mathbb{R})$

In this section, we deliver a feature of controllability of bilinear systems.

The following result establishes a necessary and sufficient condition for controllability of bilinear systems on special linear Lie group.

Proposition 3.1. *The system $\dot{x}(t) = (A + u(t)B)x(t)$, where $u(t) \in \mathbb{R}$ and $x \in \mathbb{R}^n \setminus \{0\}$, $A, B \in \mathfrak{sl}(n, \mathbb{R})$, is controllable on $SL(n, \mathbb{R})$ if only if*

$$LS(T) = \mathfrak{sl}\,(n, \mathbb{R}) \tag{3.1}$$

Proof. See [10] □

The theorem, which is presented below, is the contribution that delivers this work and its proof is based on using properties of Saturate, to finally apply Proposition 3.1.

Theorem 3.1. *We consider the bilinear control unrestricted system with simple admission, given by*

$$\dot{x}(t) = \left(A + u(t)B\right) x(t), \tag{3.2}$$

with $A, B \in \mathfrak{sl}(3, \mathbb{R})$, control $u(t) \in \mathbb{R}$ and state variable $x \in \mathbb{R}^3 \setminus \{0\}$. equivalently,

$$\Gamma = A + uB \subset \mathfrak{sl}\,(3, \mathbb{R}), \quad u \in \mathbb{R}.$$

We suppose that the matrices $A = (a_{ij})$ and B satisfy the conditions:

1. $a_{13}a_{31} < 0$

2. $\Gamma(A)$ *is strongly connected.*

3. $B = diag\,(b_1, b_2, b_3)\,, \quad b_1 < b_2 < b_3$

4. $b_i - b_j \neq b_k - b_m$ *for* $(i,j) \neq (k,m)$

Then Γ *is controllable.*

Proof. We consider the matrices A and B in the way,

$$\begin{aligned} A &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad con\, a_{13}.a_{31} < 0 \\ B &= \begin{pmatrix} -a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a-b \end{pmatrix}, \end{aligned}$$

where $\forall\,(i,j) \neq (r,s) \Rightarrow b_i - b_j \neq b_r - b_s$. In our case, $b_1 = -a$, $b_2 = b$ and $b_3 = a - b$, $a > 0$.

Without loss of generality, we assume that $a_{13} > 0$ and $a_{31} < 0$.

We show that the Lie Saturate $LS\,(\Gamma) = \mathfrak{sl}\,(3, \mathbb{R})$. (Proposition 3.1)

First of all, as $\Gamma(A)$ is strongly connected, by Theorems 2.3, A is permutation-irreducible matrix, then the system satisfies LARC.

On the other hand, as

$$\frac{A + uB}{|u|} \in LS\,(\Gamma) \Rightarrow \pm B \in LS\,(\Gamma)$$

when $u \to \pm\infty$, moreover $A \in LS\,(\Gamma)$, then, from property (2) in the Proposition 2.1 We have that

$$A_t = exp\,(t \cdot ad\,(B))\,A \in LS\,(\Gamma)\,,$$

In order to compute the matrix of the adjoint operator

$$ad\,(B) : \mathfrak{sl}(3, \mathbb{R}) \longrightarrow \mathfrak{sl}(3, \mathbb{R})$$

in a basis of $\mathfrak{sl}(3, \mathbb{R})$. We have,

$$\begin{aligned} ad\,(B)\,(E_{12}) &= -\,(a+b)\,E_{12} \\ ad\,(B)\,(E_{13}) &= (b-2a)\,E_{13} \\ ad\,(B)\,(E_{13}) &= (b-2a)\,E_{13} \\ ad\,(B)\,(E_{23}) &= (-a+2b)\,E_{23} \\ ad\,(B)\,(E_{21}) &= (a+b)\,E_{21} \\ ad\,(B)\,(E_{31}) &= (2a-b)\,E_{31} \\ ad\,(B)\,(E_{32}) &= (a-2b)\,E_{32} \\ ad\,(B)\,(E_{11} - E_{33}) &= 0 \\ ad\,(B)\,(E_{22} - E_{33}) &= 0, \end{aligned}$$

Therefore,

$$ad\,(B) = \begin{pmatrix} -a-b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b-2a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2b-a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a+b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2a-b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a-2b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

from here,

$$exp\,(t \cdot ad\,(B)) = \begin{pmatrix} e^{(-a-b)t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{(b-2a)t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{(2b-a)t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{(a+b)t} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{(2a-b)t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{(a-2b)t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

On the other hand, in the basis of $\mathfrak{sl}(3,\mathbb{R})$ we can write

$$A = \begin{pmatrix} a_{12} \\ a_{13} \\ a_{21} \\ a_{23} \\ a_{31} \\ a_{32} \\ a_{11} \\ a_{22} \end{pmatrix},$$

thus,

$$A_t \quad = \quad exp\,(t \cdot ad\,(B))\, A = \begin{pmatrix} e^{(-a-b)t}a_{12} \\ e^{(b-2a)t}a_{13} \\ e^{(2b-a)t}a_{21} \\ e^{(a+b)t}a_{23} \\ e^{(2a-b)t}a_{31} \\ e^{(a-2b)t}a_{32} \\ a_{11} \\ a_{22} \end{pmatrix}.$$

Denoted by $A_t^{7,8} = A_t - a_{11}\,(E_{11} - E_{33}) - a_{22}\,(E_{22} - E_{33})$, we have that $A_t^{7,8} \in LS\,(\Gamma)$ given that $A_t \in LS\,(\Gamma)$. Explicitly,

$$A_t^{7,8} = \begin{pmatrix} e^{(-a-b)t}a_{12} \\ e^{(b-2a)t}a_{13} \\ e^{(2b-a)t}a_{21} \\ e^{(a+b)t}a_{23} \\ e^{(2a-b)t}a_{31} \\ e^{(a-2b)t}a_{32} \\ 0 \\ 0 \end{pmatrix} \in LS\,(\Gamma).$$

From (1c) of the Proposition 2.1 we have that,

$$exp((2a-b)\,t)A_t^{7,8} = \begin{pmatrix} e^{(a-2b)t}a_{12} \\ a_{13} \\ e^{(b+a)t}a_{21} \\ e^{3at}a_{23} \\ e^{(4a-2b)t}a_{31} \\ e^{(3a-2b)t}a_{32} \\ 0 \\ 0 \end{pmatrix} \in LS\,(\Gamma)\,.$$

Now, we doing $t \to -\infty$ we obtain

$$\begin{pmatrix} 0 \\ a_{13} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in LS\,(\Gamma)\,.$$

As $a_{13} > 0$, we have that $E_{13} \in LS\,(\Gamma)$, similarly, we obtain that:

$$exp((b-2a)\,t)A_t^{7,8} = \begin{pmatrix} e^{-3at}a_{12} \\ e^{(2b-4a)t}a_{13} \\ e^{(3b-3a)t}a_{21} \\ e^{(2b-a)t}a_{23} \\ a_{31} \\ e^{(-a-b)t}a_{32} \\ 0 \\ 0 \end{pmatrix} \in LS\,(\Gamma)\,,$$

we doing $t \to \infty$ result

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_{31} \\ 0 \\ 0 \\ 0 \end{pmatrix} \in LS\,(\Gamma)\,,$$

as $a_{31} < 0$the above expression implies that $-E_{31} \in LS\,(\Gamma)$.

Now, as $LS\,(\Gamma)$ is a positive and convex cone, since $E_{13} \in LS\,(\Gamma)$ and $-E_{31} \in LS\,(\Gamma)$,it is concluded that $E_{13} - E_{31} \in LS\,(\Gamma)$.

We note that,

$$exp\,(t\,(E_{13} - E_{31})) = \begin{pmatrix} cos\,(t) & 0 & sin\,(t) \\ 0 & 1 & 0 \\ -sin\,(t) & 0 & cos\,(t) \end{pmatrix},$$

therefore, one-parameter subgroup $\{exp\,(t\,(E_{13} - E_{31})) : t \in \mathbb{R}\}$ is periodic, then, from (4) of the Proposition 2.1 implies $-\,(E_{13} - E_{31}) \in LS\,(\Gamma)$. Therefore, $\pm\,(E_{13} - E_{31}) \in LS\,(\Gamma)$.

On the other hand, from direct calculations, we obtain:

$$\pm [E_{13} - E_{31}, E_{22} - E_{33}] = \mp (E_{13} + E_{31}) \in LS(\Gamma),$$
$$\pm [E_{13} - E_{31}, E_{11} - E_{33}] = \mp 2 (E_{13} + E_{31}) \in LS(\Gamma),$$

then, $\pm E_{13}, \pm E_{31}, \pm (E_{22} - E_{33}), \pm (E_{11} - E_{33}) \in LS(\Gamma)$.

As, $\pm [E_{31}, E_{32}] = \mp E_{31} \in LS(\Gamma)$ we have that $\mp E_{32} \in LS(\Gamma)$, also,

$$\pm [E_{22} - E_{33}, E_{32}] = \pm E_{21} \in LS(\Gamma).$$

Finally, $\pm [E_{23}, E_{32}] = \mp (E_{22} - E_{33}) \in LS(\Gamma)$.

Thus, we have shown that,

$$\pm E_{12}, \pm E_{13}, \pm E_{23}, \pm E_{21}, \pm E_{31}, \pm E_{32}, \pm (E_{11} - E_{33}), \pm (E_{22} - E_{33}) \in LS(\Gamma), \quad (3.3)$$

as

$$\mathfrak{sl}(3, \mathbb{R}) = Span \{E_{12}, E_{13}, E_{23}, E_{21}, E_{31}, E_{32}, E_{11} - E_{33}, E_{22} - E_{33}\},$$

then, by (3.3) is concluded that $LS(\Gamma) = \mathfrak{sl}(3, \mathbb{R})$. □

4 Conclusion

In this paper some original contributions to the problem of controllability of bilinear systems control whose dynamics is determined by elements that lie on the Lie algebra of special linear Lie group were presented.

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Competing Interests

Author has declared that no competing interests exist.

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