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The Distribution of the Solution Zeros for the Boundary Problems of Four Points of Linear Homogeneous Differential Equations of the Sixth Order

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Short Research Article

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Abstract

In this study, the author investigates the distribution of the zeros of solutions of the sixth order linear homogenous differential equations (LHDE) with boundary conditions. An analytic approach is employed in this paper that is based on the semi critical intervals of boundary value problems. The main results are generalization of the results of LHDE of fifth order and expansion to four points boundary value problems.

Keywords: Linear differential equations; distribution of zeros for the solution; boundary value problems; semi-oscillatory interval; semi-critical interval; fundamental normal solution.

1 Introduction

The studies on distribution of zeros of solutions of linear homogeneous differential equations (LHDE) goes back to 1960s. This field of study attracts many researchers and it gains much more interest for its applications to functional equations [1,2], difference equations [3], differential equations with complex

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coefficients [4,5] differential equations with delay constant [6] and variable [7], and linear differential equations [8-19]. The research papers [8-10] adapted a geometric approach to study and prove the results of the distribution of zeros of solutions which is complicated and hard to prove because it depends on the fundamental solutions of the differential equations, so it requires more steps and sometimes needs to impose additional conditions to prove the results. Recently, an analytic approach is proposed by the authors of [11-16] for distribution of zeros of solutions. The analytic approach employs the set of normal fundamental solutions to state and prove the results by using its properties of which makes it easier to prove [17].

The authors of [11-16] investigated LHDE of the fifth order. In this paper we consider LHDE of the sixth order with four-points boundary conditions. The analytic approach is used to state and prove the properties of LHDE of the sixth order with four-points boundary conditions. Our main results in this study are $r_{3111}(\alpha) \le r_{51}(\alpha)$, $r_{2211}(\alpha) \le r_{51}(\alpha)$ and $r_{2121}(\alpha) \le r_{51}(\alpha)$.

2 Concepts and Terminology

Consider the following boundary value problem

$$x^{(6)} + \sum_{j=0}^{5} g_j(x) x^{(j)} = 0 \tag{1}$$

$$x(t_i) = \acute{x}(t_i) = \cdots = x^{(p_i)}(t_i) = 0 \tag{2}$$

where $\alpha \le t_1 < t_2 < \cdots < t_m < \infty$, m is the number of points $[\, t_i, i = 1, \dots, m]$, p_i is the number of conditions at the points $t_i, i = 1,2, \dots, m$, $g_j(x)$ are continuous on $[\propto, \beta)$

Problem (1) and (2) is called $\ll(\, p_1 p_2 \dots p_m) - \text{problem} \gg$.

When the point t_1 is fixed, the family of non-trivial solution of the problem $\ll (p_1 p_2 \dots p_m) - \text{problem} \gg$ is denoted by $W_{p_1 p_2 \dots p_m}(t, t_1)$.

Definition 2-1 [12]: The interval $[\propto, \beta), (a < \alpha < \beta < \infty)$ is called semi-oscillatory, if any non-trivial solution for equation (1) has no more than five zeros [including multiplicity] in $[\propto, \beta)$. The largest semi-oscillatory interval that begins at the point α is denoted by $\big[\alpha, r(\alpha)\big)$.

Definition 2-2 [13]: The interval $[\propto, \gamma)$ where $\ll (p_1 p_2 \dots p_m) - \text{problem} \gg$ has a unique solution is called semi-critical, and the largest semi- critical intervals that begins at the point α is denoted by $[\alpha, r_{p_1 p_2 \dots p_m}(\alpha))$.

In this research paper, we discuss non- trivial solution of boundary value problem (1) and (2) in the semi-critical intervals.

Especially $\ll (3111) - \text{problem} \gg$ where the solution has zero of multiplicity 3 at $t = t_1$, a zero of multiplicity 1 at the point $t = t_2$, a zero of multiplicity 1 at the point $t = t_3$ and a zero of multiplicity 1 at the point $t = t_4$, where $\alpha \le t_1 < t_2 < t_3 < r_{3111}(\alpha) < t_4$.

The first zero after $t = t_3$ is denoted by $r_{3111}(\alpha, t_1, t_2, t_3)$, it's clearly that

$$r_{3111}(\alpha) \le r_{3111}(\alpha, t_1, t_2, t_3) \text{ and } r_{3111}(\alpha) = \inf\ r_{3111}(\alpha, t_1, t_2, t_3). \tag{3}$$

Generally, the first zero after $t = t_{m-1}$ is denoted by $r_{p_1 p_2 \dots p_m}(\alpha, t_1, t_2, \dots, t_{m-1})$ for which

$$r_{p_1 p_2 \dots p_m}(\alpha) = \inf r_{p_1 p_2 \dots p_m}(\alpha, t_1, t_2, \dots, t_{m-1}) \tag{4}$$

where $p_1 + p_2 + \cdots + p_m = 6$.

Lemma 2-1 [14]: The function $r_{p_1 p_2 \dots p_m}(\alpha, t_1, t_2, \dots, t_{m-1})$ (where t_1 is fixed) is continuos from the right (the right limit exists) at the points $t_2, t_3, \dots, t_{m-1}$ in the set $R_{m-1}[\alpha, \infty)$. i.e.

$$\lim_{\substack{t_2 \to t_2^0 \\ t_3 \to t_3^0 \\ \vdots \\ t_{m-1} \to t_{m-1}^0}} r_{p_1 p_2 \dots p_m}(\alpha, t_1, t_2, \dots, t_{m-1}) = r_{p_1 p_2 \dots p_m}(\alpha, t_1, t_2^0, t_3^0 \dots, t_{m-1}^0).$$

Lemma 2-2 [14]: The set of fundamental normal solution for equation (1) (i.e. $\{u_j(t, t_1), j = 0,1, \dots, m-1\}$) with respect to t_1 has the following

$$u_j^{(i)}(t, t_1) = (t - t_1)^{j-i} \psi_{ij}(t, t_1), \quad i, j = 0,1, \dots, m-1. \tag{5}$$

where

$$\psi_{ij}(t_1, t_1) = \begin{cases} \frac{u_j^{(i)}(t_1, t_1)}{(j-i)!} & \text{if } \quad j \geq i \\ 0 & \text{if } \quad j < i \end{cases} \tag{6}$$

3 Main Results

In this section we present three theorems for the distribution of zeros for the solutions of LHDEs in the semi-critical intervals.

Theorem 3-1: In the interval $[\alpha, r_{3111}(\alpha))$, any non-trivial solution (for the equation (1)) that has a zero at t_1 of multiplicity five cannot have a simple zero to the right of t_1, i.e. $r_{3111}(\alpha) \leq r_{51}(\alpha)$, when $t_2 \to t_1$ *and* $t_3 \to t_1$.

Proof: First of all we show that the family of non-trivial solution for $\ll (3111) - \text{problem} \gg$ at the fixed point t_1 contains at least one solution that becomes a non-trivial solution for $\ll (51) - \text{problem} \gg$ when $t_2 \to t_1$ *and* $t_3 \to t_1$.

$$\lim_{t_3 \to t_1} \left(\lim_{t_2 \to t_1} W_{3111}(t, t_1) \right) = W_{51}(t, t_1)$$

From Vallee Poisinee theorem, for each $t_1 \in [\alpha, r(\alpha))$, there exists a semi-oscillatory interval $[t_1, r(t_1))$ ([20]). Choose $\varepsilon > 0$, such that $[t_1, t_1 + \varepsilon) \subset [t_1, r(t_1))$.

Let $u_0(t, t_1), u_1(t, t_1), \dots, u_5(t, t_1)$ be a set of fundamental normal solution for (1) with respect to t_1 i.e.

$$u_j^{(i)}(t_1, t_1) = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j \end{cases}$$

Thus, the family of non-trivial solution for the equation (1) can be written as:

$$W(t, t_1) = \sum_{j=p_1}^{5} c_j u_j(t, t_1) \tag{7}$$

where c_j is an arbitrary constant.

From the boundary condition for $\ll (3111) - \text{problem} \gg$ we get the following homogeneous system.

$$\sum_{j=p_1}^{5} c_j u_j^{(k_i)}(t_i, t_1) = 0 \tag{8}$$

where

$$k_i = 0,1, \dots, p_i - 1; \;\; i = 2,3,4; \;\; \sum_{i=1}^{m} p_i = 6$$

A necessary and sufficient condition for the system (8) to have a non-trivial solution (for unknown c_js) is:

$$D(t_1, t_2, t_3, t_4) = \det\left(u_j^{(k_i)}(t_i, t_1) : j = 3,4,5 \,; \;\; k_i = 0,1, \dots, p_i - 1 \,; \;\; i = 2,3,4\right) = 0$$

The rank of the matrix of system (8) is equal to 2 and it's different from zero.

that is

$$\triangle\,(t_1, t_2, t_3) = \begin{vmatrix} u_4(t_2, t_1) & u_5(t_2, t_1) \\ u_4(t_3, t_1) & u_5(t_3, t_1) \end{vmatrix} \neq 0$$

where $\leq t_1 < t_2 < t_3 < t_1 + \varepsilon$.

In fact, if $\triangle\,(t_1, t_2, t_3) = 0$ then the homogeneous system has a non-trivial solution $\bar{c}_4$ and $\bar{c}_5$ in $[t_1, t_1 + \varepsilon)$.Thus, the nontrivial solution for the equation (1): $W(t, t_1) = \bar{c}_4\, u_4(t, t_1) + \bar{c}_5\, u_5(t, t_1)$ has six zeros in the $[t_1, t_1 + \varepsilon)$ where $[t_1, t_1 + \varepsilon) \subset \left[t_1, r(t_1)\right)$ four of six zeros are at the point t_1, zero at the point t_2, and zero at the point t_3, this contradicts the concept of semi-oscillatory interval.

From the system (8), the first two equations constitute a system of nonhomogeneous system

$$c_4 u_4(t_2, t_1) + c_5 u_5(t_2, t_1) = -c_3 u_3(t_2, t_1)$$

$$c_4 u_4(t_3, t_1) + c_5 u_5(t_3, t_1) = -c_3 u_3(t_3, t_1)$$

Using Grammar-method, we find the values of c_4 and c_5. Note that c_3 is a free parameter depends on t_1, t_2 and t_3 that is $c_3 = c_3(t_1, t_2, t_3)$ then the general of non-trivial solution for $\ll (3111) - \text{problem} \gg$ depends on c_3 i.e.

$$W_{3111}(t, t_1) = c_3 u_3(t, t_1) + \frac{\triangle_4\,(t_1, t_2, t_3)}{\triangle\,(t_1, t_2, t_3)} u_4(t, t_1) + + \frac{\triangle_5\,\left(t_1, t_{2,t_3}\right)}{\triangle\,(t_1, t_2, t_3)} u_5(t, t_1) \tag{9}$$

where $\triangle_i\,(t_1, t_2, t_3)$, $i = 4,5$ can be obtained from $\triangle\,(t_1, t_2, t_3)$ replacing $(-c_3 u_3(t_2, t_1) \quad -c_3 u_3(t_3, t_1))^T$ by first and second columns respectively.

From the equations (5) and (9) we find

$$W_{3111}(t, t_1) = -c_3 \left(-u_3(t, t_1) + \frac{1}{(t_2 - t_1)(t_3 - t_1)} \frac{\alpha(t_1, t_2, t_3)}{\delta(t_1, t_2, t_3)} u_4(t, t_1) + \frac{1}{(t_2 - t_1)^3} \frac{\beta(t_1, t_2, t_3)}{\delta(t_1, t_2, t_3)} u_5(t, t_1) \right)$$

where

$$\delta(t_1,t_2,t_3)=\begin{vmatrix}\psi_{04}(t_2,t_1) & (t_2-t_1)\psi_{05}(t_2,t_1)\\ \psi_{04}(t_3,t_1) & (t_3-t_1)\psi_{05}(t_3,t_1)\end{vmatrix}$$

$$\alpha(t_1,t_2,t_3)=\begin{vmatrix}\psi_{03}(t_2,t_1) & (t_2-t_1)^2\psi_{05}(t_2,t_1)\\ \psi_{03}(t_3,t_1) & (t_3-t_1)^2\psi_{05}(t_3,t_1)\end{vmatrix}$$

$$\beta(t_1,t_2,t_3)=\begin{vmatrix}(t_2-t_1)\psi_{04}(t_2,t_1) & \psi_{03}(t_2,t_1)\\ (t_3-t_1)\psi_{04}(t_3,t_1) & \psi_{03}(t_3,t_1)\end{vmatrix}$$

Since c_3 is an arbitrary constant, we assume that $c_3(t_1,t_2,t_3)=(t_2-t_1)(t_3-t_1)$. By taking the limit of both sides when $t_2\to t_1$ we obtain,

$$\lim_{t_2\to t_1} W_{3111}(t,t_1)=-\frac{(\bar{t}_3-t_1)\psi_{03}(t_1,t_1)}{\psi_{04}(t_1,t_1)}u_4(t,t_1)+\frac{\psi_{04}(\bar{t}_3,t_1)\psi_{03}(t_1,t_1)}{\psi_{05}(\bar{t}_3,t_1)\psi_{04}(t_1,t_1)}u_5(t,t_1)$$

where $\bar{t}_3$ is the new position of t_3.

Now we taking the limit of both sides when $\bar{t}_3\to t_1$ we obtain,

$$\lim_{\bar{t}_3\to t_1}(\lim_{t_2\to t_1} W_{3111}(t,t_1))=\frac{\psi_{03}(t_1,t_1)}{\psi_{05}(t_1,t_1)}u_5(t,t_1) \tag{10}$$

From equation (6) we find $\psi_{03}(t_1,t_1)=\frac{1}{3!},\ \psi_{05}(t_1,t_1)=\frac{1}{5!}$. By substituting in equation (10), we find

$$\lim_{\bar{t}_3\to t_1}\left(\lim_{t_2\to t_1} W_{3111}(t,t_1)\right)=20u_5(t,t_1) \tag{11}$$

Thus we proved that the family of non-trivial solution $\ll(3111)-\text{problem}\gg$ contains a solution that becomes a solution for $\ll(51)-\text{problem}\gg$ when $t_2\to t_1$ *and* $t_3\to t_1$.

From the lemma (2-1) the function $r_{3111}(\alpha,t_1,t_2t_3)$ is continuous from the right, then we get the following inequality

$$\inf_{\propto\le t_1<t_2<t_3<r_{3111}(\propto)} r_{3111}(\alpha,t_1,t_2,t_3)\le \inf_{\propto\le t_1<r_{3111}(\propto)} r_{3111}(\alpha,t_1) \tag{12}$$

where

$$\lim_{t_3\to t_1}(\lim_{t_2\to t_1} r_{3111}(\alpha,t_1,t_2,t_3))=r_{3111}(\alpha,t_1)$$

From equations (3) and (11), we find

$$\inf_{\propto\le t_1<t_2<t_3<r_{3111}(\propto)} r_{3111}(\alpha,t_1,t_2,t_3)=r_{3111}(\alpha) \tag{13}$$

$$\inf_{\propto\le t_1<r_{3111}(\propto)} r_{3111}(\alpha,t_1)=r_{51}(\alpha) \tag{14}$$

And from (12), (13) and (14), we get $r_{3111}(\alpha)\le r_{51}(\alpha)$.

Theorem 3-2: In the interval $[\alpha, r_{2211}(\alpha))$, any non-trivial solution (for the equation (1)) that has a zero at t_1 of multiplicity five cannot have a simple zero to the right of t_1 i.e. $r_{2211}(\alpha) \le r_{51}(\alpha)$, when $t_2 \to t_1$ *and* $t_3 \to t_1$.

Proof: Form Vallee Poisinee theorem [20], (for each $t_1 \in [\alpha, r(\alpha))$, there exists a semi-oscillatory interval $[t_1, r(t_1))$. Choose $\varepsilon > 0$, such that $[t_1, t_1 + \varepsilon) \subset [t_1, r(t_1))$.

Let $u_0(t,t_1), u_1(t,t_1), \dots, u_5(t,t_1)$ be a set of fundamental normal solution for (1) with respect to t_1.

Thus, the family of non-trivial solution for the equation (1) can be written as:

$$W(t,t_1) = \sum_{j=p_1}^{5} c_j u_j(t,t_1) \tag{15}$$

From the boundary condition for $\ll (2211) - \text{problem} \gg$ we get the following homogeneous system.

$$\sum_{j=p_1}^{5} c_j u_j^{(k_i)}(t_i, t_1) = 0 \tag{16}$$

where

$$k_i = 0,1, \dots, p_i - 1; \;\; i = 2,3,4; \sum_{I=1}^{m} p_i = 6$$

A necessary and sufficient condition for the system (16) to have a non-trivial solution (for unknown $c_j^{'}s$) is:

$$D(t_1, t_2, t_3, t_4) = \det\left(u_j^{(k_i)}(t_i, t_1) : j = 2,3,4,5 \,; \;\; k_i = 0,1, \dots, p_i - 1; \;\; i = 2,3,4\right) = 0$$

The rank of the matrix of system (16) is equal to 3 and it's different from zero.

That is

$$\triangle\,(t_1, t_2, t_3) = \begin{bmatrix} u_3(t_2,t_1) & u_4(t_2,t_1) & u_5(t_2,t_1) \\ \acute{u}_3(t_2,t_1) & \acute{u}_4(t_2,t_1) & \acute{u}_5(t_2,t_1) \\ u_3(t_3,t_1) & u_4(t_3,t_1) & u_5(t_3,t_1) \end{bmatrix} \ne 0$$

where $\alpha \le t_1 < t_2 < t_3 < t_1 + \varepsilon$.

In fact, if $\triangle\,(t_1, t_2, t_3) = 0$ then the homogeneous system has a non-trivial solution $\bar{c}_3$, $\bar{c}_4$ and $\bar{c}_5$ in $[t_1, t_1 + \varepsilon)$. Thus, the nontrivial solution for the equation (1), $W(t,t_1) = \bar{c}_3\, u_3(t,t_1) + \bar{c}_4\, u_4(t,t_1) + \bar{c}_5\, u_5(t,t_1)$ has six zeros in the semi-oscillatory interval $[t_1, t_1 + \varepsilon)$ where $[t_1, t_1 + \varepsilon) \subset \left[t_1, r(t_1)\right)$ three of six zeros are at the point t_1, two zeros at the point t_2, and zero at the point t_3, this contradicts the concept of semi-oscillatory interval.

In the system (16), the first three equations constitute a system of nonhomogeneous equation

$$c_3 u_3(t_2,t_1) + c_4 u_4(t_2,t_1) + c_5 u_5(t_2,t_1) = -c_2 u_2(t_2,t_1)$$

$$c_3 \acute{u}_3(t_2,t_1) + c_4 \acute{u}_4(t_2,t_1) + c_5 \acute{u}_5(t_2,t_1) = -c_2 \acute{u}_2(t_2,t_1)$$

$$c_3 u_3(t_3,t_1) + c_4 u_4(t_3,t_1) + c_5 u_5(t_3,t_1) = -c_2 u_2(t_3,t_1)$$

Using Grammar-method, we find the values of c_3, c_4 and c_5. Note that c_2 is a free parameter depends on t_1, t_2 and t_3 that is $c_2 = c_2(t_1, t_2, t_3)$ then the family of non-trivial solution for $\ll (2211) - \text{problem} \gg$ depends on c_2 i.e.

$$W_{2211}(t, t_1) = c_2 u_2(t, t_1) + \sum_{i=3}^{5} \frac{\Delta_i\,(t_1, t_2, t_3)}{\Delta\,(t_1, t_2, t_3)} u_i(t, t_1) \tag{17}$$

where $\Delta_i\,(t_1, t_2, t_3)$, (i = 3,4,5) can be obtained from $\Delta\,(t_1, t_2, t_3)$ replacing $(-c_2 u_2(t_2, t_1) \quad -c_2 \acute{u}_2(t_2, t_1) \quad -c_2 u_2(t_3, t_1))^T$ by first, second and third columns respectively.

From the equation (5) we find

$$\Delta\,(t_1, t_2, t_3) = (t_3 - t_1)^3 (t_2 - t_1)^5 d(t_1, t_2, t_3)$$

$$\Delta_3\,(t_1, t_2, t_3) = (t_3 - t_1)^2 (t_2 - t_1)^4 d_3(t_1, t_2, t_3)$$

$$\Delta_4\,(t_1, t_2, t_3) = (t_3 - t_1)^2 (t_2 - t_1)^4 d_4(t_1, t_2, t_3)$$

$$\Delta_5\,(t_1, t_2, t_3) = (t_3 - t_1)^2 (t_2 - t_1)^3 d_5(t_1, t_2, t_3)$$

Where

$$d(t_1, t_2, t_3) = (t_2 - t_1)^3 \psi_{03}(t_3, t_1) \varphi(t_2, t_1) + \delta(t_1, t_2, t_3)$$

$$d_3(t_1, t_2, t_3) = (t_2 - t_1)^4 \psi_{02}(t_3, t_1) \alpha(t_2, t_1) + \delta_3(t_1, t_2, t_3)$$

$$d_4(t_1, t_2, t_3) = -(t_2 - t_1)^3 \psi_{02}(t_3, t_1) \beta(t_2, t_1) + \delta_4(t_1, t_2, t_3)$$

$$d_5(t_1, t_2, t_3) = -(t_2 - t_1)^3 \psi_{02}(t_3, t_1) \gamma(t_2, t_1) + \delta_5(t_1, t_2, t_3)$$

$$\varphi(t_2, t_1) = \begin{vmatrix} \psi_{04}(t_2, t_1) & \psi_{05}(t_2, t_1) \\ \psi_{14}(t_2, t_1) & \psi_{15}(t_2, t_1) \end{vmatrix}, \alpha(t_1, t_2) = \begin{vmatrix} \psi_{04}(t_2, t_1) & \psi_{05}(t_2, t_1) \\ \psi_{14}(t_2, t_1) & \psi_{15}(t_2, t_1) \end{vmatrix}$$

$$\beta(t_1, t_2) = \begin{vmatrix} \psi_{03}(t_2, t_1) & \psi_{05}(t_2, t_1) \\ \psi_{13}(t_2, t_1) & \psi_{15}(t_2, t_1) \end{vmatrix}, \gamma(t_1, t_2) = \begin{vmatrix} \psi_{03}(t_2, t_1) & \psi_{04}(t_2, t_1) \\ \psi_{13}(t_2, t_1) & \psi_{14}(t_2, t_1) \end{vmatrix}$$

$$\delta(t_1, t_2, t_3) \to 0 \; and \; \delta_i(t_1, t_2, t_3) \to 0, \;\; (i = 3,4,5\,) \;\; when \; t_2 \to t_1.$$

By substituting in equation (17), we find

$$\begin{aligned} W_{2211}(t, t_1) = -c_2 \Bigg(& -u_2(t, t_1) + \frac{1}{(t_2 - t_1)(t_3 - t_1)} \frac{d_3(t_1, t_2, t_3) + \delta_3(t_1, t_2, t_3)}{d(t_1, t_2, t_3) + \delta(t_1, t_2, t_3)} u_3(t, t_1) \\ & + \frac{1}{(t_2 - t_1)(t_3 - t_1)} \frac{d_4(t_1, t_2, t_3) + \delta_4(t_1, t_2, t_3)}{d(t_1, t_2, t_3) + \delta(t_1, t_2, t_3)} u_4(t, t_1) \\ & + \frac{1}{(t_3 - t_1)(t_2 - t_1)^2} \frac{d_5(t_1, t_2, t_3) + \delta_5(t_1, t_2, t_3)}{\delta(t_1, t_2, t_3) + \delta(t_1, t_2, t_3)} u_5(t, t_1) \Bigg) \end{aligned}$$

Since c_2 is an arbitrary constant, we assume that $c_2(t_1, t_2, t_3) = -(t_2 - t_1)^2 (t_3 - t_1)$. By taking the limit of both sides when $t_2 \to t_1$ we obtain,

$$\lim_{t_2\to t_1} W_{2211}(t,t_1) = \frac{\psi_{02}(\bar{t}_3,t_1)\gamma(t_1,t_1)}{\psi_{03}(\bar{t}_3,t_1)\varphi(t_1,t_1)} u_5(t,t_1) \tag{18}$$

Where $\bar{t}_3$ is the new position of t_3.

now taking the limit of both sides when $\bar{t}_3 \to t_1$ we obtain,

$$\lim_{\bar{t}_3\to t_1}\left(\lim_{t_2\to t_1} W_{2211}(t,t_1)\right) = \frac{\psi_{02}(t_1,t_1)\gamma(t_1,t_1)}{\psi_{03}(t_1,t_1)\varphi(t_1,t_1)} u_5(t,t_1) \tag{19}$$

and from equation (6) we find

$\psi_{02}(t_1,t_1) = \frac{1}{2}$, $\psi_{03}(t_1,t_1) = \frac{1}{6}$, $\alpha(t_1,t_1) = \frac{1}{144}$ and $\varphi(t_1,t_1) = \frac{1}{2880}$. By substituting in equation (19), we find

$$\lim_{\bar{t}_3\to t_1}\left(\lim_{t_2\to t_1} W_{2211}(t,t_1)\right) = 60u_5(t,t_1) \tag{20}$$

Thus we proved that the family of non-trivial solution $\ll (2211) - \text{problem} \gg$ contains a solution that becomes a solution for $\ll (51) - \text{problem} \gg$ when $t_2 \to t_1$ *and* $t_3 \to t_1$.

From the lemma (2-1) the function $r_{2211}(\alpha,t_1,t_2,t_3)$ is continuous from the right, then we get the following inequality

$$\inf_{\propto\le t_1<t_2<t_3<r_{2211}(\propto)} r_{2211}(\alpha,t_1,t_2,t_3) \le \inf_{\propto\le t_1<r_{2211}(\propto)} r_{2211}(\alpha,t_1) \tag{21}$$

where

$$\lim_{t_2\to t_1}(\lim_{t_2\to t_1} r_{2211}(\alpha,t_1,t_2,t_3)) = r_{2211}(\alpha,t_1)$$

From equations (4) and (20), we find

$$\inf_{\propto\le t_1<t_2<t_3<r_{2211}(\propto)} r_{2211}(\alpha,t_1,t_2,t_3) = r_{2211}(\alpha) \tag{22}$$

$$\inf_{\propto\le t_1<r_{2211}(\propto)} r_{2211}(\alpha,t_1) = r_{51}(\alpha) \tag{23}$$

From (21), (22) and (23), we get $r_{2211}(\alpha) \le r_{51}(\alpha)$.

Theorem 3-3: In the interval $[\alpha, r_{2121}(\alpha))$, any non-trivial solution (for the equation (1)) that has a zero at t_1 of multiplicity five cannot have a simple zero to the right of t_1 i.e. $r_{2121}(\alpha) \le r_{51}(\alpha)$, when $t_2 \to t_1$ *and* $t_3 \to t_1$.

Proof: Form Vallee Poisinee theorem ([20]), for each $t_1 \in [\alpha, r(\alpha))$, there exists a semi-oscillatory interval $[t_1, r(t_1))$. Choose $\varepsilon > 0$, such that $[t_1, t_1 + \varepsilon) \subset [t_1, r(t_1))$.

Let $u_0(t,t_1), u_1(t,t_1), \ldots, u_5(t,t_1)$ be a set of fundamental normal solution for (1) with respect to t_1.

Thus, the family of non-trivial solution for the equation (1) can be written as:

$$W(t,t_1)=\sum_{j=p_1}^{5} c_j u_j(t,t_1) \tag{24}$$

From the boundary condition for $\ll (2121)-$ problem $\gg$ we get the following homogeneous system.

$$\sum_{j=p_1}^{5} c_j u_j^{(k_i)}(t_i,t_1)=0 \tag{25}$$

where

$$k_i=0,1,\dots,p_i-1;\quad i=2,3,4\,;\ \sum_{I=1}^{m} p_i=6$$

A necessary and sufficient condition for the system (25) to have a non-trivial solution (for unknown c_j's) is:

$$D(t_1,t_2,t_3,t_4)=\det\left(u_j^{(k_i)}(t_i,t_1): j=2,3,4,5\,;\quad k_i=0,1,\dots,p_i-1\,;\ i=2,3,4\right)=0$$

The rank of the matrix of system (25) is equal to 3 and it's different from zero.

That is

$$\Delta\,(t_1,t_2,t_3)=\begin{bmatrix} u_3(t_2,t_1) & u_4(t_2,t_1) & u_5(t_2,t_1)\\ u_3(t_3,t_1) & u_4(t_3,t_1) & u_5(t_3,t_1)\\ \acute{u}_3(t_3,t_1) & \acute{u}_4(t_3,t_1) & \acute{u}_5(t_3,t_1)\end{bmatrix}\neq 0$$

where $\alpha \le t_1 < t_2 < t_3 < t_1+\varepsilon$.

In fact, if $\Delta\,(t_1,t_2,t_3)=0$ then the homogeneous system has a non-trivial solution $\bar{c}_3$, $\bar{c}_4$ and $\bar{c}_5$ in $[t_1,t_1+\varepsilon)$. Thus, the nontrivial solution for the equation (1), $W(t,t_1)=\bar{c}_3\,u_3(t,t_1)+\bar{c}_4\,u_4(t,t_1)+\bar{c}_5\,u_5(t,t_1)$ has six zeros in the semi-oscillatory interval $[t_1,t_1+\varepsilon)$ where $[t_1,t_1+\varepsilon)\subset[t_1,r(t_1))$ three of six zeros are at the point t_1, zero at the point t_2, and two zeros at the point t_3, this contradicts the concept of semi-oscillatory interval.

In the system (25), the first three equations constitute a system of nonhomogeneous system

$$c_3u_3(t_2,t_1)+c_4u_4(t_2,t_1)+c_5u_5(t_2,t_1)=-c_2u_2(t_2,t_1)$$

$$c_3u_3(t_3,t_1)+c_4u_4(t_3,t_1)+c_5u_5(t_3,t_1)=-c_2u_2(t_3,t_1)$$

$$c_3\acute{u}_3(t_3,t_1)+c_4\acute{u}_4(t_3,t_1)+c_5\acute{u}_5(t_3,t_1)=-c_2\acute{u}_2(t_3,t_1)$$

Using Grammar-method, we find the values of c_3, c_4 and c_5. Note that c_2 is a free parameter depends on t_1, t_2 and t_3 that is $c_2=c_2(t_1,t_2,t_3)$ then the family of non-trivial solution for $\ll (2121)-$ problem $\gg$ depends on c_2 i.e.

$$W_{2121}(t,t_1)=c_3u_2(t,t_1)+\sum_{i=3}^{5}\frac{\Delta_i\,(t_1,t_2,t_3)}{\Delta\,(t_1,t_2,t_3)}u_i(t,t_1) \tag{26}$$

Where $\Delta_i\ (t_1, t_2, t_3), (\mathrm{i} = 3,4,5)$ can be obtained from $\Delta\ (t_1, t_2, t_3)$ replacing $(-c_2 u_2(t_2, t_1) \quad -c_2 u_2(t_3, t_1) \quad -c_2 \acute{u}_2(t_2, t_1))^{\mathrm{T}}$ by first, second and third columns respectively.

From the equation (5) we find

$$\Delta\ (t_1, t_2, t_3) = (t_3 - t_1)^5 (t_2 - t_1)^3 d(t_1, t_2, t_3)$$

$$\Delta_3\ (t_1, t_2, t_3) = (t_3 - t_1)^4 (t_2 - t_1)^2 d_3(t_1, t_2, t_3)$$

$$\Delta_4\ (t_1, t_2, t_3) = (t_3 - t_1)^3 (t_2 - t_1)^2 d_4(t_1, t_2, t_3)$$

$$\Delta_5\ (t_1, t_2, t_3) = (t_3 - t_1)^3 (t_2 - t_1)^2 d_5(t_1, t_2, t_3)$$

Where

$$d(t_1, t_2, t_3) = (t_3 - t_1)^3 \psi_{03}(t_2, t_1)\varphi(t_3, t_1) + \delta(t_1, t_2, t_3)$$

$$d_3(t_1, t_2, t_3) = -(t_3 - t_1)^4 \psi_{02}(t_2, t_1)\alpha(t_3, t_1) + \delta_3(t_1, t_2, t_3)$$

$$d_4(t_1, t_2, t_3) = (t_3 - t_1)^4 \psi_{02}(t_2, t_1)\beta(t_3, t_1) + \delta_4(t_1, t_2, t_3)$$

$$d_5(t_1, t_2, t_3) = -(t_3 - t_1)^3 \psi_{02}(t_2, t_1)\gamma(t_3, t_1) + \delta_5(t_1, t_2, t_3)$$

$$\varphi(t_3, t_1) = \begin{vmatrix} \psi_{04}(t_3, t_1) & \psi_{05}(t_3, t_1) \\ \psi_{14}(t_3, t_1) & \psi_{15}(t_3, t_1) \end{vmatrix}, \ \alpha(\mathrm{t}_3, \mathrm{t}_1) = \begin{vmatrix} \psi_{04}(t_3, t_1) & \psi_{05}(t_3, t_1) \\ \psi_{14}(t_3, t_1) & \psi_{15}(t_3, t_1) \end{vmatrix}$$

$$\beta(\mathrm{t}_3, \mathrm{t}_2) = \begin{vmatrix} \psi_{03}(t_3, t_1) & \psi_{05}(t_3, t_1) \\ \psi_{13}(t_3, t_1) & \psi_{15}(t_3, t_1) \end{vmatrix}, \ \gamma(\mathrm{t}_3, \mathrm{t}_2) = \begin{vmatrix} \psi_{03}(t_3, t_1) & \psi_{04}(t_3, t_1) \\ \psi_{13}(t_3, t_1) & \psi_{14}(t_3, t_1) \end{vmatrix}$$

$$\delta(t_1, t_2, t_3) \to 0 \ and \ \delta_i(t_1, t_2, t_3) \to 0 \ , (i = 3,4,5 \ ) \quad when \ t_2 \to t_1.$$

By substituting in equation (26), we obtain

$$\begin{aligned} \mathrm{W}_{2121}(\mathrm{t}, \mathrm{t}_1) = -\mathrm{c}_2 \Bigg(& -\mathrm{u}_2(\mathrm{t}, \mathrm{t}_1) + \frac{1}{(t_2 - t_1)(t_3 - t_1)} \frac{\mathrm{d}_3(t_1, t_2, t_3) + \delta_3(t_1, t_2, t_3)}{\mathrm{d}(t_1, t_2, t_3) + \delta(t_1, t_2, t_3)} \mathrm{u}_3(\mathrm{t}, \mathrm{t}_1) \\ & - \frac{1}{(t_2 - t_1)(t_3 - t_1)^2} \frac{\mathrm{d}_4(t_1, t_2, t_3) + \delta_4(t_1, t_2, t_3)}{\mathrm{d}(t_1, t_2, t_3) + \delta(t_1, t_2, t_3)} \mathrm{u}_4(\mathrm{t}, \mathrm{t}_1) \\ & + \frac{1}{(t_2 - t_1)(t_3 - t_1)^2} \frac{d_5(t_1, t_2, t_3) + \delta_5(t_1, t_2, t_3)}{\delta(t_1, t_2, t_3) + \delta(t_1, t_2, t_3)} u_5(t, t_1) \Bigg) \end{aligned}$$

Since c_2 is an arbitrary constant, we assume that $\mathrm{c}_2(t_1, t_2, t_3) = -(t_2 - t_1)(\mathrm{t}_3 - \mathrm{t}_1)^2$. By taking the limit of both sides when $\mathrm{t}_2 \to \mathrm{t}_1$ we obtain,

$$\begin{aligned} \lim_{\mathrm{t}_2 \to \mathrm{t}_1} \mathrm{W}_{2121}(\mathrm{t}, \mathrm{t}_1) = & \frac{(\bar{t}_3 - t_1)^2 \psi_{02}(t_1, t_1)\alpha(\bar{t}_3, t_1)}{\psi_{03}(t_1, t_1)\varphi(\bar{t}_3, t_1)} u_3(t, t_1) - \\ & \frac{(\bar{t}_3 - t_1)\ \psi_{02}(t_1, t_1)\beta(\bar{t}_3, t_1)}{\psi_{03}(t_1, t_1)\varphi(\bar{t}_3, t_1)} u_4(t, t_1) + \frac{\psi_{02}(t_1, t_1)\gamma(\bar{t}_3, t_1)}{\psi_{03}(t_1, t_1)\varphi(\bar{t}_3, t_1)} u_5(t, t_1) \end{aligned} \quad (27)$$

Where $\bar{t}_3$ is the new position of t_3.

Now taking the limit of both sides when $\bar{t}_3 \to t_1$ we obtain,

$$\lim_{\bar{t}_3\to t_1}\left(\lim_{t_2\to t_1} W_{2121}(t,t_1)\right) = \frac{\psi_{02}(t_1,t_1)\gamma(t_1,t_1)}{\psi_{03}(t_1,t_1)\varphi(t_1,t_1)} u_5(t,t_1) \tag{28}$$

From equation (6) we find

$\psi_{02}(t_1,t_1) = \frac{1}{2}$, $\psi_{03}(t_1,t_1) = \frac{1}{6}$, $\gamma(t_1,t_1) = \frac{1}{144}$ and $\varphi(t_1,t_1) = \frac{1}{2880}$. By substituting in equation (28), we find

$$\lim_{\bar{t}_3\to t_1}\left(\lim_{t_2\to t_1} W_{2121}(t,t_1)\right) = 60u_5(t,t_1) \tag{29}$$

Thus we proved that the family of non-trivial solution $\ll (2121) - \text{problem} \gg$ contains a solution that becomes a solution for $\ll (51) - \text{problem} \gg$ when $t_2 \to t_1$ *and* $t_3 \to t_1$.

By the lemma (2-1) the function $r_{2121}(\alpha, t_1, t_2, t_3)$ is continuous from the right, then we get the following inequality

$$\inf_{\propto \le t_1 < t_2 < t_3 < r_{2121}(\propto)} r_{2121}(\alpha, t_1, t_2, t_3) \le \inf_{\propto \le t_1 < r_{2121}(\propto)} r_{2121}(\alpha, t_1) \tag{30}$$

where

$$\lim_{t_3\to t_1}\left(\lim_{t_2\to t_1} r_{2121}(\alpha, t_1, t_2, t_3)\right) = r_{2121}(\alpha, t_1)$$

From equations (4) and (29), we find

$$\inf_{\propto \le t_1 < t_2 < t_3 < r_{2121}(\propto)} r_{2121}(\alpha, t_1, t_2, t_3) = r_{2121}(\alpha) \tag{31}$$

$$\inf_{\propto \le t_1 < r_{2121}(\propto)} r_{2121}(\alpha, t_1) = r_{51}(\alpha) \tag{32}$$

from (30), (31) and (32), we get $r_{2121}(\alpha) \le r_{51}(\alpha)$.

4 Conclusion

This study is an investigation of the distribution of zeros of the solutions of 6th order DE with boundary conditions. Theorems 3-1, 3-2 and 3-3 state that the semi-critical intervals $r_{3111}(\alpha)$, $r_{2211}(\alpha)$ and $r_{2121}(\alpha)$ are less than or equal to the semi-critical intervals $r_{51}(\alpha)$ *when* $t_2 \to t_1$ *and* $t_3 \to t_1$. Therefore we conclude that $max\{\ r_{3111}(\alpha),\ \ r_{2211}(\alpha)\ ,\ \ r_{2121}(\alpha)\ \} \le r_{51}(\alpha)$.

Competing Interests

Author has declared that no competing interests exist.

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