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A Useful Result on the Covariance Between Ito Integrals

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

This article introduces a general result on the covariance between two Ito integrals driven by two different Brownian motions, which slightly generalizes the isometry property. This result finds applications in mathematical finance, e.g. it enables to determine the probability distribution of the integrated interest rate process in exponential-affine models of the yield curve.

Keywords: Ito integral; multidimensional Brownian motion; interest rate process; yield curve model.

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1 Introduction

Let $\{W_t, t \geq 0\}$ be a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. One of the main properties of the Ito integral is the isometry, which states that, for any function $Y\left(\omega \in \Omega, t \in \mathbb{R}^+\right)$ such that the integral $\int_0^t Y_s dW_s$ exists, $\forall 0 \leq t < \infty$, the second order moment of

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the integral is given by :

$$E\left[\left(\int_0^t Y_s dW_s\right)^2\right] = \int_0^t E\left[Y_s^2\right] ds \tag{1}$$

A minor generalization is to consider the covariation between $\int_0^t Y_s^{(1)} dW_s$ and $\int_0^t Y_s^{(2)} dW_s$. As long as the functions $Y^{(1)}$ and $Y^{(2)}$ are both predictable with respect to the same filtration, it is an elementary extension to obtain :

$$\text{cov}\left[\int_0^t Y_s^{(1)} dW_s, \int_0^t Y_s^{(2)} dW_s\right] = \int_0^t E\left[Y_s^{(1)} Y_s^{(2)}\right] ds \tag{2}$$

This is a consequence of the fact that, in general, any isometric mapping between two real normed inner product spaces preserves inner products.

However, further results on the covariance between Ito integrals may be needed in stochastic models involving multiple random factors with non-zero correlation. Indeed, considering each random factor in time as an Ito integral, the integrands we are then dealing with are no longer functions of the same Brownian motion, and they are not functions of independent Brownian motions either, i.e. we are confronted with the problem of computing :

$$\text{cov}\left[\int_0^{t_1} Y_s^{(1)} dW_s^{(1)}, \int_0^{t_2} Y_s^{(2)} dW_s^{(2)}\right], (t_1, t_2) \in \mathbb{R}_+^2 \tag{3}$$

where $Y^{(1)}$ and $Y^{(2)}$ may be functions of both Brownian motions $W^{(1)}$ and $W^{(2)}$, with $d\left[W^{(1)}, W^{(2)}\right](t) \neq 0$

In this paper, a solution of this problem is given under conditions that remain not too restrictive for a fairly large number of applications, allowing notably for random integrands. This result turns out to be useful for applications of stochastic calculus to finance. In particular, it is instrumental in performing analytical calculations related to yield curve models such as the exponential affine one, which is widely used in the research departments of large financial institutions around the world.

2 General Results

Let $\left\{W_t^{(1)}, t \geq 0\right\}$ and $\left\{W_t^{(2)}, t \geq 0\right\}$ be two standard Brownian motions with correlation coefficient ρ.

The integrals $\int_0^{t_1} Y_s^{(1)} dW_s^{(1)}$ and $\int_0^{t_2} Y_s^{(2)} dW_s^{(2)}$, $(t_1, t_2) \in \mathbb{R}_+^2$, are defined under the condition that $Y^{(1)}$ and $Y^{(2)}$ are square integrable and predictable. This raises the question of the existence of a unique filtration with respect to which both $Y^{(1)}$ and $Y^{(2)}$ would be measurable. That filtration would not simply be the product of the natural filtrations of each Brownian motion, as is the case for multidimensional Brownian motion, because of the non-independence of $W^{(1)}$ and $W^{(2)}$. It would have to be an expanded filtration. The subject of the expansion of filtrations began to be actively investigated at the end of the previous century by a paper of K. Itô himself [1] which showed that if $\{W_t, t \geq 0\}$is a standard Brownian motion, then one can expand the natural filtration $\mathcal{F}_t$ of W_t by adding the $\sigma-$algebra generated by the random variable W_1 to all $\mathcal{F}_t$ of the filtration. It was followed by a number of papers (see, e.g., [2], [3], [4]) dealing with the conditions under which semimartingales remain semimartingales in a filtration expanded by a random variable or a stochastic process. A survey of this literature can be found in [5]. Since our correlated Brownian motions $W^{(1)}$ and $W^{(2)}$are defined on spaces equipped with a probability measure, i.e. a $\sigma-$ finite measure, the existence of an expanded filtration for the pair $\left(W^{(1)}, W^{(2)}\right)$ is guaranteed by the Hahn–Kolmogorov theorem ([6]). However, a rigorous use of such a filtration would require a redefinition of some of the tools used when working with the natural filtrations of $W^{(1)}$ and $W^{(2)}$. For example, it is well-known that, if $W^{(1)}$ and $W^{(2)}$are independent, then the process defined by the product $W^{(1)}W^{(2)}$ is a martingale. But this cannot be proven simply by conditioning $E\left[W_{t+s}^{(1)}W_{t+s}^{(2)}\right]$, $\forall s>0$, on either the natural filtration $\mathcal{F}_t^{(1)}$of $W^{(1)}$or the natural filtration $\mathcal{F}_t^{(2)}$ of $W^{(2)}$. Indeed, either conditioning is performed with respect to $\mathcal{F}_t^{(1)}$, which yields:

$$\begin{gathered} E\left[W_{t+s}^{(1)}W_{t+s}^{(2)}\middle|\mathcal{F}_t^{(1)}\right]= \\ \mathrm{E}\left[W_{t+s}^{(1)}\middle|\mathcal{F}_t^{(1)}\right]E\left[W_{t+s}^{(2)}\middle|\mathcal{F}_t^{(1)}\right]=W_t^{(1)}E\left[W_{t+s}^{(2)}\middle|\mathcal{F}_t^{(1)}\right]\neq W_t^{(1)}W_t^{(2)} \end{gathered} \tag{4}$$

or conditioning is performed with respect to $\mathcal{F}_t^{(2)}$, which yields:

$$\begin{gathered} E\left[W_{t+s}^{(1)}W_{t+s}^{(2)}\middle|\mathcal{F}_t^{(2)}\right]= \\ \mathrm{E}\left[W_{t+s}^{(1)}\middle|\mathcal{F}_t^{(2)}\right]E\left[W_{t+s}^{(2)}\middle|\mathcal{F}_t^{(2)}\right]=E\left[W_{t+s}^{(1)}\middle|\mathcal{F}_t^{(2)}\right]W_t^{(2)}\neq W_t^{(1)}W_t^{(2)} \end{gathered} \tag{5}$$

Whereas if one assumes the existence of a larger filtration $\mathcal{F}_t^{(12)}$with respect to which both $W^{(1)}$ and $W^{(2)}$are simultaneously measurable, and such that the properties verified under $\mathcal{F}_t^{(1)}$ and $\mathcal{F}_t^{(2)}$ are also verified under $\mathcal{F}_t^{(12)}$, then one immediately obtains:

$$E\left[W_{t+s}^{(1)}W_{t+s}^{(2)}\middle|\mathcal{F}_t^{(12)}\right]=E\left[W_{t+s}^{(1)}\middle|\mathcal{F}_t^{(12)}\right]E\left[W_{t+s}^{(2)}\middle|\mathcal{F}_t^{(12)}\right]=W_t^{(1)}W_t^{(2)} \tag{6}$$

But (6) is wrong because, although the Brownian motions $W^{(1)}$ and $W^{(2)}$ are martingales with respect to their natural filtrations $\mathcal{F}_t^{(1)}$ and $\mathcal{F}_t^{(2)}$, respectively, they are not martingales with respect to the larger filtration $\mathcal{F}_t^{(12)}$.

Since the purpose of this paper is not to address deeps questions of the general theory, but to provide a simple computational tool that slightly generalizes the isometry property of the Ito integral, we will circumvent the need for an expansion of filtration, i.e. we will refer exclusively to the natural filtrations of $W^{(1)}$ and $W^{(2)}$, by appropriately restricting the class of permissible Ito integrals. Fortunately, such a restriction still allows to solve a number of practical problems, as will be illustrated in Section 3.

We can now state our general result.

Theorem

Let $\left\{W_t^{(1)}, t\geq 0\right\}$ and $\left\{W_t^{(2)}, t\geq 0\right\}$ be two standard Brownian motions with correlation coefficient ρ.

The natural filtrations generated by $\left\{W_t^{(1)}, t\geq 0\right\}$ and $\left\{W_t^{(2)}, t\geq 0\right\}$ are denoted by $\mathcal{F}_t^{(1)}$ and $\mathcal{F}_t^{(2)}$, respectively. Let $Y_t^{(1)}=f\left(t, W_t^{(1)}\right)$ and $Y_t^{(2)}=g\left(t, W_t^{(2)}\right)$, where f and g are non-anticipating, right-continuous real functions with left limits.

Then, $\forall 0\leq t\leq T$, we have :

$$\operatorname{cov}\left[\int_0^t Y_s^{(1)}dW_s^{(1)}, \int_0^T Y_s^{(2)}dW_s^{(2)}\right]=\rho\int_0^t E\left[Y_s^{(1)}Y_s^{(2)}\right]ds \tag{7}$$

Remark 1 : the fact that no expansion of filtration is required relies on the restriction that the function $Y_t^{(1)}$ may depend on $W_t^{(1)}$ but not on $W_t^{(2)}$
Remark 2 : the result still holds if the functions $Y_t^{(1)}$and $Y_t^{(2)}$ are left-continuous with right limits

Proof

The function f is continuous and $\mathcal{F}_t^{(1)}-$adapted, while the function g is continuous and $\mathcal{F}_t^{(2)}-$adapted. Therefore, the process $\left\{Y_t^{(1)}, t \geq 0\right\}$ is predictable with respect to $\mathcal{F}_t^{(1)}$ and the process $\left\{Y_t^{(2)}, t \geq 0\right\}$ is predictable with respect to$\mathcal{F}_t^{(2)}$. Moreover, both $\left\{Y_t^{(1)}, t \geq 0\right\}$ and $\left\{Y_t^{(2)}, t \geq 0\right\}$ are bounded on any closed time interval $[0, T]$ and thus square integrable. Hence, the integrals $I_1(t) = \int\limits_0^t Y_s^{(1)} dW_s^{(1)}$ and $I_2(t) = \int\limits_0^t Y_s^{(2)} dW_s^{(2)}$ are Ito integrals, with $I_1(t)$ being an $\mathcal{F}_t^{(1)}-$martingale with zero expectation and $I_2(t)$ being an $\mathcal{F}_t^{(2)}-$martingale with zero expectation, and we have:

$$\begin{aligned}&\operatorname{cov}\left[I_1(t), I_2(T)\right]\\ &= E\left[\int\limits_0^t Y_s^{(1)} dW_s^{(1)} \int\limits_0^T Y_s^{(2)} dW_s^{(2)}\right] = E\left[\int\limits_0^t Y_s^{(1)} dW_s^{(1)} \left(\int\limits_0^t Y_s^{(2)} dW_s^{(2)} + \int\limits_t^T Y_s^{(2)} dW_s^{(2)}\right)\right]\end{aligned} \tag{8}$$

By conditioning with respect to $\mathcal{F}_t^{(1)}$, and by using the $\mathcal{F}_t^{(1)}-$measurability of $I_1(t)$ as well as the independence of $\int\limits_t^T Y_s^{(2)} dW_s^{(2)}$ with respect to $\mathcal{F}_t^{(1)}$, we obtain :

$$E\left[\int\limits_0^t Y_s^{(1)} dW_s^{(1)} \int\limits_t^T Y_s^{(2)} dW_s^{(2)}\right] = E\left[E\left[\int\limits_0^t Y_s^{(1)} dW_s^{(1)} \int\limits_t^T Y_s^{(2)} dW_s^{(2)} \middle| \mathcal{F}_t^{(1)}\right]\right] \tag{9}$$

$$= E\left[\int\limits_0^t Y_s^{(1)} dW_s^{(1)} E\left[\int\limits_t^T Y_s^{(2)} dW_s^{(2)} \middle| \mathcal{F}_t^{(1)}\right]\right] = E\left[\int\limits_0^t Y_s^{(1)} dW_s^{(1)} E\left[\int\limits_t^T Y_s^{(2)} dW_s^{(2)}\right]\right] = 0 \tag{10}$$

Notice that the independence of $\int\limits_t^T Y_s^{(2)} dW_s^{(2)}$ with respect to $\mathcal{F}_t^{(1)}$does not derive from any independence between the processes $\{I_1(t), t \geq 0\}$ and $\{I_2(t), t \geq 0\}$, since $\left\{W_t^{(1)}, t \geq 0\right\}$ and $\left\{W_t^{(2)}, t \geq 0\right\}$ have non-zero correlation, but from the Markov property of $\left\{W_t^{(2)}, t \geq 0\right\}$.

Since $\left\{Y_t^{(1)}, t \geq 0\right\}$ and $\left\{Y_t^{(2)}, t \geq 0\right\}$ are right-continuous, square integrable processes with left limits, they can be approximated, respectively, by the following simple processes:

$$\tilde{Y}_t^{(1)} = \sum_{i=0}^{n-1} \phi_i^{(1)} \mathbb{I}_{\left(t_i, t_{i+1}\right]}(t) \tag{11}$$

$$\tilde{Y}_t^{(2)} = \sum_{i=1}^{n-1} \phi_i^{(2)} \mathbb{I}_{\left(t_i, t_{i+1}\right]}(t) \tag{12}$$

where :
- n is the number of points in a sequence of partitions of $[0 = t_0, t = t_n]$
- $\phi_i^{(1)}$ and $\phi_i^{(2)}$ are real constants if $\left(Y^{(1)}, Y^{(2)}\right)$ is a pair of deterministic processes
- $\phi_i^{(1)}$ and $\phi_i^{(2)}$are square integrable real random variables if $\left(Y^{(1)}, Y^{(2)}\right)$ is a pair of random processes;
the $\phi_i^{(1)\prime}$s are then $\mathcal{F}_{t_i}^{(1)}-$measurable, while the $\phi_i^{(2)\prime}$s are $\mathcal{F}_{t_i}^{(2)}-$measurable
- $\phi_{t_0}^{(1)} = Y_{t_0}^{(1)}$ and $\phi_{t_0}^{(2)} = Y_{t_0}^{(2)}$

It is a fundamental result from the theory of continuous-time processes that the integrals $I_1(t)$ and $I_2(t)$ can be defined as the following limits :

$$I_1(t) = \lim_{n\to\infty}\sum_{i=0}^{n-1}\phi_i^{(1)}\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right) \tag{13}$$
$$I_2(t) = \lim_{n\to\infty}\sum_{i=0}^{n-1}\phi_i^{(2)}\left(W_{t_{i+1}}^{(2)} - W_{t_i}^{(2)}\right)$$

where convergence is in mean square [7] . Hence, as $n \to \infty$,

$$E\left[I_1(t)I_2(t)\right] = E\left[\sum_{i=0}^{n-1}\phi_i^{(1)}\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\sum_{j=0}^{n-1}\phi_j^{(2)}\left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\right]$$

$$= \sum_{i=0}^{n-1}E\left[\phi_i^{(1)}\phi_i^{(2)}\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{i+1}}^{(2)} - W_{t_i}^{(2)}\right)\right] + \sum_{i=0}^{n-1}\sum_{\substack{j=0\\ j\neq i}}^{n-1}E\left[\phi_i^{(1)}\phi_j^{(2)}\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\right]$$

$$\triangleq S_1 + S_2 \tag{14}$$

By conditioning on $\mathcal{F}_{t_i\wedge t_j}^{(1)}$, we have:

$$S_2 = \sum_{i=0}^{n-1}\sum_{\substack{j=0\\ j\neq i}}^{n-1}E\left[E\left[\phi_i^{(1)}\phi_j^{(2)}\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\middle|\mathcal{F}_{t_i\wedge t_j}^{(1)}\right]\right] \tag{15}$$

$$= \sum_{i=0}^{n-1}\sum_{\substack{j=0\\ j\neq i}}^{n-1}E\left[\phi_i^{(1)}E\left[\phi_j^{(2)}\middle|\mathcal{F}_{t_i}^{(1)}\right]E\left[\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\right]\right]\mathbb{I}_{\{t_i<t_j\}} \tag{16}$$

$$+\sum_{i=0}^{n-1}\sum_{\substack{j=0\\ j\neq i}}^{n-1}E\left[E\left[\phi_i^{(1)}\phi_j^{(2)}\middle|\mathcal{F}_{t_j}^{(1)}\right]E\left[\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\right]\right]\mathbb{I}_{\{t_j<t_i\}} \tag{17}$$

Denoting by $\left\{\bar{W}_t^{(2)}, t \geq 0\right\}$ a standard Brownian motion independent of $\left\{W_t^{(1)}, t \geq 0\right\}$ defined on the same filtered probability space as $\left\{W_t^{(2)}, t \geq 0\right\}$, the following orthogonal decomposition of $W^{(2)}$ with respect to $W^{(1)}$ holds :

$$W_t^{(2)} = \rho W_t^{(1)} + \sqrt{1-\rho^2}\bar{W}_t^{(2)} \tag{18}$$

Therefore,

$$E\left[\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\right]$$
$$= E\left[\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(\rho W_{t_{j+1}}^{(1)} + \sqrt{1-\rho^2}\bar{W}_{t_{j+1}}^{(2)} - \rho W_{t_j}^{(1)} - \sqrt{1-\rho^2}\bar{W}_{t_j}^{(2)}\right)\right] \tag{19}$$
$$= \rho\mathrm{cov}\left[W_{t_{i+1}}^{(1)}, W_{t_{j+1}}^{(1)}\right] - \rho\mathrm{cov}\left[W_{t_{i+1}}^{(1)}, W_{t_j}^{(1)}\right] - \rho\mathrm{cov}\left[W_{t_i}^{(1)}, W_{t_{j+1}}^{(1)}\right] + \rho\mathrm{cov}\left[W_{t_i}^{(1)}, W_{t_j}^{(1)}\right] \tag{20}$$

If i and j are two natural integers such that $i < j$, then $\sup_i ((i+1)-j) = 0$; similarly, if $j < i$, then $\sup_j ((j+1)-i) = 0$. Thus,

$$E\left[\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\right]$$

$$= \rho\left(t_{i+1} - t_{i+1} - t_i + t_i\right)\mathbb{I}_{\{i<j\}} + \rho\left(t_{j+1} - t_j - t_{j+1} + t_j\right)\mathbb{I}_{\{i>j\}} = 0 \quad (21)$$

Hence, $S_2 = 0$.

Next, by conditioning w.r.t. $\mathcal{F}_{t_i}^{(1)}$, we have :

$$S_1 = \sum_{i=0}^{n-1} E\left[E\left[\phi_i^{(1)}\phi_i^{(2)}\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{i+1}}^{(2)} - W_{t_i}^{(2)}\right)\middle|\mathcal{F}_{t_i}^{(1)}\right]\right] \quad (22)$$

$$= \sum_{i=0}^{n-1} E\left[\phi_i^{(1)} E\left[\phi_i^{(2)}\middle|\mathcal{F}_{t_i}^{(1)}\right] E\left[\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(\rho W_{t_{i+1}}^{(1)} + \sqrt{1-\rho^2}\bar{W}_{t_{i+1}}^{(2)} - \rho W_{t_i}^{(1)} - \sqrt{1-\rho^2}\bar{W}_{t_i}^{(2)}\right)\right]\right] \quad (23)$$

$$= \sum_{i=0}^{n-1} E\left[\phi_i^{(1)} E\left[\phi_i^{(2)}\middle|\mathcal{F}_{t_i}^{(1)}\right]\rho\left(t_{i+1} - t_i\right)\right] \quad (24)$$

$$= \rho\sum_{i=0}^{n-1} E\left[\phi_i^{(1)} E\left[\phi_i^{(2)}\middle|W_{t_i}^{(1)}\right]\right]\left(t_{i+1} - t_i\right) \quad (25)$$

$$= \rho\sum_{i=0}^{n-1} E\left[\phi_i^{(1)} E\left[\phi_i^{(2)}\middle|f\left(W_{t_i}^{(1)}\right)\right]\right]\left(t_{i+1} - t_i\right) \quad (26)$$

$$= \rho\sum_{i=0}^{n-1} E\left[\phi_i^{(1)} E\left[\phi_i^{(2)}\middle|\phi_i^{(1)}\right]\right]\left(t_{i+1} - t_i\right) \quad (27)$$

$$= \rho\sum_{i=0}^{n-1} E\left[\phi_i^{(1)}\phi_i^{(2)}\right]\left(t_{i+1} - t_i\right) \quad (28)$$

where we have used the Markov property of $\left\{Y_t^{(2)}, t \geq 0\right\}$ to go from (24) to (25) and the non-anticipating nature of function f to go from (25) to (26).

The last sum in (28) converges in mean square to $\rho\int_0^t E\left[Y_s^{(1)}Y_s^{(2)}\right] ds$, which completes the proof.

Remark : the computation of the sum S_1 is shorter, but somewhat heuristic, if one notices that the non-zero correlation between $\left\{W_t^{(2)}, t \geq 0\right\}$ and $\left\{W_t^{(1)}, t \geq 0\right\}$ implies that $\mathcal{F}_{t_i}^{(1)} \subset \mathcal{F}_{t_i}^{(2)}$, so that the $\phi_i^{(1)\prime}$s are not only $\mathcal{F}_{t_i}^{(1)}$−measurable but also $\mathcal{F}_{t_i}^{(2)}$−measurable; thus, by conditioning w.r.t. $\mathcal{F}_{t_i}^{(2)}$ instead of $\mathcal{F}_{t_i}^{(1)}$, we obtain :

$$S_1 = \sum_{i=0}^{n-1} E\left[E\left[\phi_i^{(1)}\phi_i^{(2)}\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{i+1}}^{(2)} - W_{t_i}^{(2)}\right)\middle|\mathcal{F}_{t_i}^{(2)}\right]\right] \quad (29)$$

$$= \sum_{i=0}^{n-1} E\left[\phi_i^{(1)}\phi_i^{(2)} E\left[\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(\rho W_{t_{i+1}}^{(1)} + \sqrt{1-\rho^2}\bar{W}_{t_{i+1}}^{(2)} - \rho W_{t_i}^{(1)} - \sqrt{1-\rho^2}\bar{W}_{t_i}^{(2)}\right)\right]\right] \quad (30)$$

$$(31)$$

$$= \rho\sum_{i=0}^{n-1} E\left[\phi_i^{(1)}\phi_i^{(2)}\right]\left(t_{i+1} - t_i\right)$$

3 Application to Mathematical Finance

We now turn to an application of the general result of Section 2 to the modeling of the yield curve and the valuation of fixed income instruments in mathematical finance.

Let the short-term interest rate be driven by the following three-factor, time-dependent, mean-reverting stochastic differential equation:

$$dr_t = a\left(b\left(t\right) - r_t\right)dt + \sigma_{R1} dW_t^{(1)} + \sigma_{R2} dW_t^{(2)} + \sigma_{R3} dW_t^{(3)} \tag{32}$$

where $(a, \sigma_{R1}, \sigma_{R2}, \sigma_{R3}) \in \mathbb{R}_+^4$ and $b\left(t\right)$ is a deterministic function of t satisfying a linear growth condition so that there exists a solution to eq. (32) [8]. The processes $\left\{W_t^{(1)}, t \geq 0\right\}$ $\left\{W_t^{(2)}, t \geq 0\right\}$ and $\left\{W_t^{(3)}, t \geq 0\right\}$ are three standard Brownian motions with $d\left[W^{(1)}, W^{(2)}\right](t) = \rho_{1.2}dt$, $d\left[W^{(1)}, W^{(3)}\right](t) = \rho_{1.3}dt$ and $d\left[W^{(2)}, W^{(3)}\right](t) = \rho_{2.3}dt$.

This is an extended Vasicek model [9], belonging to the class of exponential affine models, which are extensively used in the financial markets [10]. Compared to the original Vasicek model, two Brownian motion have been added. This is because statistical studies of the yield curve have consistently pointed out the need to introduce several random factors in order to reproduce the observed variability of market rates. More specifically, empirical studies have shown that three correlated Brownian motions are enough to capture over 95% of the actual yield curve [11]. In this respect, σ_{R1}, σ_{R2} and σ_{R3} are the sensitivities of the interest rate to the first, the second and the third random factors, respectively, affecting the yield curve. Only a three-factor model of the rate process such as the one given by eq. (32) is capable of reproducing the observed changes in the shape of the yield curve such as the way it alternately steepens and flattens over time, its inversion on certain maturities or its various humps and peaks. This is an improvement over existing two-factor models such as the G2++ model of Brigo and Mercurio [11], or the Hull-White model [12], which generate curves that are too smooth and regular. The mean-reversion feature in eq. (32) is also validated by empirical data. It has been made time-dependent, so that calibration to the current market prices can be achieved by fitting an appropriate function $b\left(t\right)$.

To compute discounted asset prices, whether they are fixed-income or equity, it is necessary to compute the integral $\int_0^T r_t dt$, i.e. the integrated interest rate process. By taking the stochastic differential of $\exp\left(at\right) r_t$ and integrating it on $[0, t]$, eq. (32) is found to admit the following solution:

$$r_t = r_0 e^{-at} + a \int_0^t e^{-a(t-s)} b\left(s\right) ds + \sigma_{R1} \int_0^t e^{-a(t-s)} dW_s^{(1)} + \sigma_{R2} \int_0^t e^{-a(t-s)} dW_s^{(2)} + \sigma_{R3} \int_0^t e^{-a(t-s)} dW_s^{(3)} \tag{33}$$

At any given time $t \geq 0$, the interest rate process is the sum of a constant, a Cauchy-Riemann integral and several Ito integrals with non-random integrands. Hence, it is normally distributed, and so is the integral $\int_0^T r_t dt$ too. Notice, though, that the strong statement of the property of stability of the Laplace-Gauss distribution with respect to addition is required, as the random variables that are added together in eq. (33) are not independent from one another. Applying a generalized, stochastic Fubini's theorem [13], as well as the zero expectation property of Ito integrals, one easily gets the moment of order 1 :

$$E\left[\int_0^T r_t dt\right] \triangleq \bar{\mu}_r = \frac{r_0}{a}\left(1-e^{-aT}\right) + a\int_0^T \left(\int_0^t e^{-a(t-u)} b(u)\, du\right) dt \tag{34}$$

Since var $\left[\int_0^T r_t dt\right] = \int_0^T \int_0^t \mathrm{cov}\left[r_t, r_s\right] dsdt$, our main result in Section 1 is instrumental in deriving the moment of order 2 of the integrated rate process. Applying eq. (7) yields :

$$\mathrm{cov}\left[r_t, r_s\right] = e^{-a(s+t)}\left(\frac{e^{2as}-1}{2a}\right)\left(\sigma_{R1}^2 + \sigma_{R2}^2 + \sigma_{R3}^2 + 2\rho_{1.2}\sigma_{R1}\sigma_{R2} + 2\rho_{1.3}\sigma_{R1}\sigma_{R3} + 2\rho_{2.3}\sigma_{R2}\sigma_{R3}\right) \tag{35}$$

so that the variance of the time integral of $\{r(t), t \geq 0\}$ is equal to :

$$\mathrm{var}\left[\int_0^T r_t dt\right] = \frac{1}{a^2}\left(\sigma_{R1}^2 + \sigma_{R2}^2 + \sigma_{R3}^2 + 2\rho_{1.2}\sigma_{R1}\sigma_{R2} + 2\rho_{1.3}\sigma_{R1}\sigma_{R3} + 2\rho_{2.3}\sigma_{R2}\sigma_{R3}\right)\left(T - \frac{2}{a}\left(1-e^{-aT}\right) + \frac{1-e^{-2aT}}{2a}\right) \tag{36}$$

It can be shown that the Brownian motion $W^{(3)}$ admits the following orthogonal decomposition [14] :

$$W_t^{(3)} = \rho_{1.3} W_t^{(1)} + \rho_{2.3|1} \bar{W}_t^{(2)} + \sigma_{3|1.2} \bar{W}_t^{(3)} \tag{37}$$

where $W^{(1)}$, $\bar{W}^{(2)}$ and $\bar{W}^{(3)}$ are pairwise independent, standard Brownian motions, and where the following definitions hold:

$$\sigma_{2|1} = \sqrt{1-\rho_{1.2}^2} \tag{38}$$

$$\rho_{2.3|1} = \frac{\rho_{2.3} - \rho_{1.2}\rho_{1.3}}{\sigma_{2|1}} \qquad \sigma_{3|1.2} = \sqrt{1-\rho_{1.3}^2 - \rho_{2.3|1}^2} \tag{39}$$

Thus, for fixed $T \geq 0$, the following equality in law can be established :

$$\int_0^T r_t\, dt = \bar{\mu}_r + \bar{\sigma}_r\left(\left(\sigma_{R1} + \sigma_{R2}\rho_{1.2} + \sigma_{R3}\rho_{1.3}\right) W_T^{(1)} + \left(\sigma_{R2}\sigma_{2|1} + \sigma_{R3}\rho_{2.3|1}\right)\bar{W}_T^{(2)} + \sigma_{R3}\sigma_{3|1.2}\bar{W}_T^{(3)}\right) \tag{40}$$

where

$$\bar{\sigma}_r = \frac{1}{a\sqrt{T}}\sqrt{T - \frac{2}{a}\left(1-e^{-aT}\right) + \frac{1}{2a}\left(1-e^{-2aT}\right)} \tag{41}$$

It is well-known in mathematical finance that zero-coupon bonds are the building blocks for the construction of the yield curve. Let us denote by $B(t,T)$ the value, at a given time time $t \geq 0$, of a zero-coupon bond maturing at time $T \geq t$. By definition, $B(t,T) = E\left[\exp\left(-\int_t^T r_s ds\right)\right]$. Thus, the exact value of $B(t,T)$ is readily obtained through the formula for the moment generating function of a normally distributed random variable :

$$B(t,T) = E\left[\exp\left(-\int_t^T r_s ds\right)\right] = \exp\left(-A(t,T) r_t + C(t,T)\right),\ 0 \leq t \leq T \tag{42}$$

$$A(t,T) = \frac{1}{a}\left(1-e^{-a(T-t)}\right) \tag{43}$$

$$C(t,T) = -a\int_t^T \left(\int_t^u e^{-a(u-s)} b(s)\, ds\right) du \tag{44}$$

$$+\frac{1}{a^2}\left(\sigma_{R1}^2+\sigma_{R2}^2+\sigma_{R3}^2+2\rho_{1.2}\sigma_{R1}\sigma_{R2}+2\rho_{1.3}\sigma_{R1}\sigma_{R3}+2\rho_{2.3}\sigma_{R2}\sigma_{R3}\right)\left(T+\frac{2\left(e^{-a(T-t)}-1\right)}{a}+\frac{1-e^{-2a(T-t)}}{2a}\right)$$

Last but not least, one can also obtain in closed form the probability distribution of an equity asset whose instantaneous variations are driven by the following four-dimensional geometric Brownian motion:

$$dS_t = r_t S_t dt + \sigma_S S_t dW_t^{(4)} \tag{45}$$

In eq. (45), σ_S is a positive real constant and $W^{(4)}$ is a standard Brownian motion having non-zero correlation with the Brownian motions $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$, thus allowing to model covariation between fixed-income and equity asset classes. If we denote by $\rho_{i.j}$ the correlation coefficient between $W^{(i)}$ and $W^{(j)}$, $(i,j) \in \{1,2,3,4\}$, the Brownian motion $W^{(4)}$ admits the following orthogonal decomposition [14]:

$$W_t^{(4)} = \rho_{1.4} W_t^{(1)} + \rho_{2.4|1}\bar{W}_t^{(2)} + \rho_{3.4|1.2}\bar{W}_t^{(3)} + \sigma_{4|1.2.3}\bar{W}_t^{(4)} \tag{46}$$

where the four Brownian motions $W^{(1)}$, $\bar{W}^{(2)}$, $\bar{W}^{(3)}$ and $\bar{W}^{(4)}$ are independent from one another, and where the following definitions apply:

$$\rho_{2.4|1} = \frac{\rho_{2.4}-\rho_{1.2}\rho_{1.4}}{\sigma_{2|1}} \tag{47}$$

$$\rho_{3.4|1.2} = \frac{\rho_{3.4}-\rho_{1.3}\rho_{1.4}-\rho_{2.3|1}\,\rho_{2.4|1}}{\sigma_{3|1.2}} \tag{48}$$

$$\sigma_{4|1.2.3} = \sqrt{1-\rho_{1.4}^2-\rho_{2.4|1}^2-\rho_{3.4|1.2}^2} \tag{49}$$

Then, by applying Ito's lemma to $\ln(S_t/S_0)$, by using the representation of $\int_0^T r_t dt$ provided by eq. (40) and the decomposition of $W^{(4)}$ provided by eq. (46), one can obtain the following strong solution to eq. (45):

$$S_t = S_0 \exp\begin{pmatrix} \left(\bar{\mu}_r - \frac{\sigma_S^2}{2}\right)t + W_t^{(1)}\left(\bar{\sigma}_r\left(\sigma_{R1}+\sigma_{R2}\rho_{1.2}+\sigma_{R3}\rho_{1.3}\right)+\sigma_S\rho_{1.4}\right) \\ +\bar{W}_T^{(2)}\left(\bar{\sigma}_r\left(\sigma_{R2}\sigma_{2|1}+\sigma_{R3}\rho_{2.3|1}\right)+\sigma_S\rho_{2.4|1}\right) \\ +\bar{W}_T^{(3)}\left(\bar{\sigma}_r\sigma_{R3}\sigma_{3|2.1}+\sigma_S\rho_{3.4|1.2}\right)+\bar{W}_T^{(4)}\sigma_S\sigma_{4|1.2.3} \end{pmatrix} \tag{50}$$

Since eq. (45) describes the variations of an equity asset price process under the risk-neutral measure, eq. (50) allows to derive closed form formulae for equity option prices in a general setting where equity and fixed-income assets are correlated and the yield curve is driven by three factors. As a consequence of the theory of non-arbitrage valuation of contingent claims ([15], [16]), the prices, at the current time denoted by t_0, of European call or put options written on S with expiry T, can be expressed as linear combinations of expectations of two kinds:

$$E_Q\left[\exp\left(-\int_0^T r_t dt\right) f(S_T)\,\mathbb{I}\{\mathcal{Z}\}\right] \tag{51}$$

$$E_Q\left[\exp\left(-\int_0^T r_t dt\right)\mathbb{I}\{\mathcal{Z}\}\right] \tag{52}$$

In (51) and (52), $\mathbb{I}_{\{\mathcal{Z}\}}$denotes the indicator function, $\mathcal{Z}$ represents an event involving S_T that has to occur for the option to have strictly positive value at expiry T, while f is a specific function implied by the payoff under consideration, and Q is the equivalent martingale measure under which the variations of S are driven by eq. (45). To compute expectations of the first kind, such as given by

(51), one needs to switch from the original risk-neutral measure Q to the measure usually known as forward-neutral, under which the process $B^{(4)}$ defined by $B_t^{(4)} = W_t^{(4)} + \sigma_S t$ is a standard Brownian motion. This change of measure is no different than the one required when the interest rate is assumed to be constant in the standard Black-Scholes model. The computation of expectations of the second kind such as given by (51), however, requires a new change of measure. Applying Ito's lemma to $\ln(B(t,T))$ under Q, and then integrating on $[0,t]$, one can obtain:

$$B(t,T) = B(0,T)\,\beta_t\,L(t,T) \tag{53}$$

where :

β_t is the money market account, defined by $\beta_t = \left\{\exp\left(\int_0^t r_s ds\right), t \geq 0\right\}$

and :

$$L(t,T) = \exp\begin{pmatrix} -\frac{1}{2}\int_0^t \left(A^2(s,T)\left(\sigma_{R1}^2 + \sigma_{R2}^2 + \sigma_{R3}^2 + 2\rho_{1.2}\sigma_{R1}\sigma_{R2} + 2\rho_{1.3}\sigma_{R1}\sigma_{R3} + 2\rho_{2.3}\sigma_{R2}\sigma_{R3}\right)\right) ds \\ -(\sigma_{R1} + \sigma_{R2}\rho_{1.2} + \sigma_{R3}\rho_{1.3})\int_0^t A(s,T)\,dW_s^{(1)} - \left(\sigma_{R2}\sigma_{2|1} + \sigma_{R3}\rho_{2.3|1}\right)\int_0^t A(s,T)\,d\bar{W}_s^{(2)} \\ -\sigma_{R3}\sigma_{3|1.2}\int_0^t A(s,T)\,d\bar{W}_s^{(3)} \end{pmatrix} \tag{54}$$

Hence we have :

$$E_Q\left[\exp\left(-\int_0^T r_t dt\right)\mathbb{I}\{\mathcal{Z}\}\right] = E_Q\left[\frac{B(t,T)}{\beta_t}\mathbb{I}\{\mathcal{Z}\}\right] = B(0,T)\,E_{P_{B_T}}\left[\mathbb{I}\{\mathcal{Z}\}\right] \tag{55}$$

where P_{B_T} is the equivalent martingale measure under which the numeraire is the zero-coupon bond, whose Radon-Nikodym derivative is given by:

$$\frac{dP_{B_T}}{dQ}\,|\mathcal{F}_t = L(t,T) \tag{56}$$

The following processes :

$$B_t^{(1)} = W_t^{(1)} + (\sigma_{R1} + \sigma_{R2}\rho_{1.2} + \sigma_{R3}\rho_{1.3})\int_0^t A(s,t)\,ds = W_t^{(1)} + \frac{(\sigma_{R1} + \sigma_{R2}\rho_{1.2} + \sigma_{R3}\rho_{1.3})\left(1 - e^{-aT} - T\right)}{a^2} \tag{57}$$

$$\bar{B}_t^{(2)} = \bar{W}_t^{(2)} + \left(\sigma_{R2}\sigma_{2|1} + \sigma_{R3}\rho_{2.3|1}\right)\int_0^t A(s,t)\,ds = \bar{W}_t^{(2)} + \frac{\left(\sigma_{R2}\sigma_{2|1} + \sigma_{R3}\rho_{2.3|1}\right)\left(1-e^{-aT}-T\right)}{a^2} \tag{58}$$

$$\bar{B}_t^{(3)} = \bar{W}_t^{(3)} + \sigma_{R3}\sigma_{3|1.2}\int_0^t A(s,t)\,ds = \bar{W}_t^{(3)} + \frac{\sigma_{R3}\sigma_{3|1.2}\left(1-e^{-aT}-T\right)}{a^2} \tag{59}$$

are standard Brownian motions under the measure P_{B_T}

The analytical tractability of the three-factor model of the rate process given by eq. (32), leading to closed form solutions for bond and option prices, is an attractive feature for practitioners for two reasons. First, it enables them to avoid having to resort to Monte Carlo simulation, which is a relatively slow and inaccurate method. This is an advantage over the so-called "market models" which do not preserve analytical tractability [11]. Furthermore, it allows easy and fast calibration procedures. In particular, since a swaption can be decomposed into a portfolio of options on zero coupon bonds, a calibration to swaption market prices is made possible. Details about how this can be carried out are given in [17].

4 Conclusions

This article has shown that the covariance between Ito integrals in a multidimensional Brownian motion model with cross-correlation can be dealt with by referring exclusively to the natural filtrations of one-dimensional Brownian motions, albeit at the expense of some restriction of the class of permissible integrands. Although the applications described in this paper only involve deterministic integrands of the Ito integrals, it must be noticed that the general result given in Section 2 extends to random integrands.

Competing Interests

Author has declared that no competing interests exist.

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