Vibration of Prestressed Isotropic Rectangular Plate with General Boundary Conditions and Under Partially Distributed Loads Moving at Varying Velocities

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Authors' contributions

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Abstract

The dynamic analysis of prestressed rectangular plate with general boundary conditions and under partially distributed loads moving at varying velocities is investigated in this paper. A procedure involving the generalized two-dimensional integral transform with beam functions as kernel of transformation is used to reduce the governing fourth order partial differential equation to a second order coupled ordinary differential equation. The Struble's asymptotic technique is then used to simplify this equation to make it amenable to the methods of integral transformation and convolution theory. By means of these, the analytical solution valid for all variants of classical boundary conditions to the dynamical problem is obtained. The analytical solution and numerical analysis show that the critical speed for the moving distributed mass problem is reached earlier than that of the moving distributed force problem for both illustrative examples considered. The results further show that an upward variations of foundation stiffness K, rotatory inertia correction factor R^0, subgrade modulus G and axial force N decrease the response amplitude of the rectangular plate. Finally, for fixed pertinent structural parameters, the transverse displacement response of the rectangular plate under moving

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partially distributed forces is not an upper bound to the case when subjected to moving partially distributed masses. Hence, safety is not guaranteed for a design based on the moving partially distributed force problem.

Keywords: Prestressed rectangular plate; general boundary conditions; integral transformation; critical speed.

1 Introduction

This paper is sequel to an earlier one by Oni and Ogunbamike [1] that considered the response to distributed loads moving at varying velocities of an elastic isotropic rectangular plate resting on Pasternak foundation. In particular, this paper is a generalization of the theory advanced in [1]. The study of plate flexure under moving loads forms a very important structural element in Engineering design and construction. It has also become the objective of various investigations in the field of applied Mathematics and Physics. In general, problems of this type are mathematically complex when the inertial effect of the moving load is taken into consideration Fryba [2], The first major breakthrough in this field of research was the work of Stanisic et al. [3] who solved the problem of simply supported non-Mindlin plate under a multi-masses moving system by making use of an approximation of the Dirac delta function. Only the inertia terms that measure the effect of local acceleration in the direction of the deflection was considered. The method of solutions was based on the Fourier Sine transform technique. The solutions so obtained were shown to converge very rapidly. The work of Stanisic et al. was taken up much later by Gbadeyan and Oni [4] who investigated the dynamic analysis of an elastic plate continuously supported by an elastic Pasternak foundation and traversed by an arbitrary number of concentrated masses. All the components of the inertial terms were considered and the rectangular plate was taken to be simply supported. Huang and Thambiratnam [5] in a similar manner studied isotropic homogeneous elastic rectangular plate resting on an elastic Winkler foundation under a single concentrated load. Finite strip method was employed. Numerical examples show that when the load moves with zero or initial velocity, the dynamic response of the structure is steady and unlike the response due to the sudden application of a load. Worthy of note, also, is the work of Shadnam et al. [6] who investigated the dynamics of plates under the influence of relatively large masses, moving along an arbitrary trajectory on the plate surface. As an example, the dynamic response of a rectangular plate, simply supported on all its edges, under a mass moving parallel to one of its sides and also travelling along a circular trajectory is presented by means of operational calculus. Analysis shows that the response of structures due to moving mass, which have often been neglected in the past, must be properly taken into account because it often differs significantly from the moving force model. Amiri et al. [7] investigated the elastodynamic response of a rectangular Mindlin plate under concentrated moving loads as well as other arbitrarily selected distribution area of loads. Closed form solution for the moving force load case was derived using direct separation of variable and eigenfunction expansion method. A semi-analytical solution was presented for moving mass load case. Although the aforementioned investigations involving concentrated loads are impressive, they do not represent the physical reality of the problem formulation as concentrated masses do not exist physically. in practice moving loads are in the form of moving distributed masses which are actually distributed over a small segment or over the entire length of the structural member they traverse Andi et al. [8]. Research works on two-dimensional structural members traversed by distributed loads are scanty. Researchers in this are Dada [9], Gbadeyan and Dada [10] and Isede and Gbadeyan [11] to mention a few. More recently, Gbadeyan and Dada [12] used a finite difference algorithm to investigate the elastodynamic response of a Mindlin plate subjected to a distributed moving mass. The simply supported edge condition was used as an illustrative example. It was found that the maximum shearing forces, bending and twisting moments occur almost at the same time. It is noted in these works that numerical simulations are adopted as analytical techniques are herculean. In a vibrating system such as this, it is pertinent to treat the phenomena of resonance which are not revealed through numerical simulations. And as such, analytical solutions are desirable as solutions so obtained often shed more light on vital information about the vibrating system. To this end, Andi and Oni [13] undertook the dynamic behaviour of an elastic isotropic rectangular plate under travelling distributed loads. The method of solution is purely analytical. The solution procedure

is based on the two dimensional Finite Fourier sine transform to transform the governing fourth order partial differential equation to a coupled second order ordinary differential equation which is later simplified by the asymptotic method of Struble to make it amenable to simpler Laplace transform and convolution theory. Other researchers who worked on this include Wu [14] Andi and Oni [15], Oni [16], Oni and Ogunyebi [17], to mention but a few. It is however noted that these authors considered dynamical system in which the traversing loads move with constant velocities. Infact, the more practical cases when the velocities at which these loads moves are no longer constants but vary with time have received but little attention in literature. Thus, this paper therefore investigates the flexural vibrations of prestressed rectangular plate under partially distributed loads with general boundary conditions. Both gravity and inertia effects of the uniformly distributed masses are taken into consideration, and the plate is taken ton rest on Pasternak foundation. The solution technique which is analytical and suitable for all variants of classical boundary conditions involves using the generalized two-dimensional integral transform, the expression of Heaviside function as a Fourier series and the use of modified Struble's asymptotic technique to solve the problem of the dynamical system.

2 Governing Equations

Consider the dynamic transverse displacement $V(x, y, t)$ of the mid-surface of a prestressed isotropic rectangular plate of span L_y along the y-axis and span L_x along the x-axis carrying a partially distributed mass moving with variable velocity c along a straight line $y = y_0$ parallel to the x-axis.
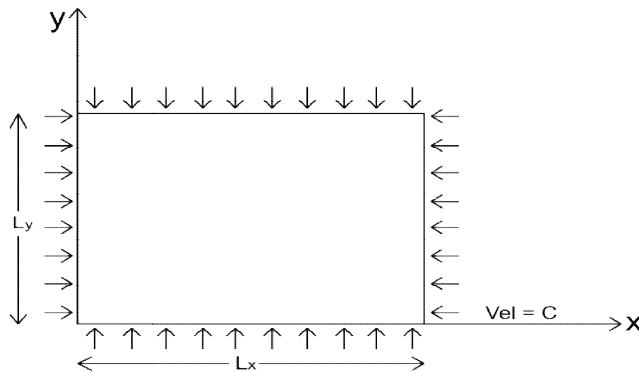

Fig. 1. Rectangle plate subject to uniform distributed loads

If the plate model is resting on a Pasternak foundation, $V(x, y, t)$ is governed by the fourth order partial differential equation given by

$$\left[D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 - \mu R^0 \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 \right] V(x, y, t) - \left[N_x \frac{\partial^2 V(x, y, t)}{\partial x^2} + N_y \frac{\partial^2 V(x, y, t)}{\partial y^2} \right]$$

$$+ \mu \frac{\partial^2 V(x, y, t)}{\partial t^2} + KV(x, y, t) - G\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 V(x, y, t) = P_F(x, y, t)\left[1 - \frac{\Delta^*}{g}\left[V(x, y, t) \right] \right] \quad (1)$$

E is the young modulus, v is the Poisson's ratio $(v < 1)$, μ is the mass of the plate per unit length, x is the position coordinate in x-direction, y is the position coordinate in y-direction, t is the time, h is the plate thickness, K is the foundation stiffness, G is the shear modulus and R^0 is the measure of rotatory inertia,,

$D = \dfrac{Eh^3}{12(1-v)}$ is the bending rigidity of the plate and is constant throughout the plate, $P_F(x,y,t)$ is the continuous moving force which travels from point $y = y_0$ on the plate along a straight line parallel to the x-axis, g is the acceleration due to gravity.

In view of equations (2), (3), (4) and (5), equation (1) can be written as [1]

$$
\begin{aligned}
&D\left[\frac{\partial^4 V(x,y,t)}{\partial x^4} + 2\frac{\partial^4 V(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 V(x,y,t)}{\partial y^4}\right] + \mu\frac{\partial^2 V(x,y,t)}{\partial t^2} - N_x\frac{\partial^2 V(x,y,t)}{\partial x^2} - N_y\frac{\partial^2 V(x,y,t)}{\partial y^2} \\
&- \mu R^0\left[\frac{\partial^4 V(x,y,t)}{\partial x^2 \partial t^2} + \frac{\partial^4 V(x,y,t)}{\partial y^2 \partial t^2}\right] + KV(x,y,t) - G\left[\frac{\partial^2 V(x,y,t)}{\partial x^2} + \frac{\partial^2 V(x,y,t)}{\partial y^2}\right] \\
&+ MH\left[x - \left(x_0 + ct + \tfrac{1}{2}at^2\right)\right]H\left[y - y_0\right]\left[(c+at)^2\frac{\partial^2 V(x,y,t)}{\partial x^2} + \frac{\partial^2 V(x,y,t)}{\partial t^2}\right. \\
&\left.+ 2(c+at)\frac{\partial^2 V(x,y,t)}{\partial x \partial t} + a\frac{\partial V(x,y,t)}{\partial x}\right] = MgH\left[x - \left(x_0 + ct + \tfrac{1}{2}at^2\right)\right]H\left[y - y_0\right]
\end{aligned}
\tag{2}
$$

Equation (2) is the fourth order partial differential equation governing the flexural motion of a prestressed isotropic rectangular plate on Pasternak foundation under the action of uniform partially distributed loads moving at non-uniform velocity. The boundary conditions are taken to be arbitrary, while the initial conditions without any loss of generality are given by

$$
V(x,y,t) = 0 = \frac{\partial V(x,y,t)}{\partial t}
\tag{3}
$$

3 Solution Technique

The analysis of the response to a moving partially distributed mass of isotropic rectangular plate resting on a Pasternak foundation and subjected to arbitrary boundary conditions is carried out in this section by employing the solution technique already alluded to [4, 5, 6, 7]. In particular the generalized two-dimensional integral is defined as

$$
\tilde{V}(j,k,t) = \int_0^{L_x}\int_0^{L_y} V(x,y,t)V_j(x)V_k(y)\,dx\,dy
\tag{4}
$$

with the inverse

$$
V(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\mu}{U_j}\frac{\mu}{U_k}\tilde{V}(j,k,t)iV_j(x)V_k(y)
\tag{5}
$$

where

$$
U_j = \int_0^{L_x}\mu V_j^2(x)\,dx \; ; \qquad U_k = \int_0^{L_x}\mu V_k^2(y)\,dy
\tag{6}
$$

where $V_j(x)$ and $V_k(y)$ are respectively the beam functions in x and y directions defined respectively as

$$
V_j(x) = \sin\frac{\lambda_j x}{L_x} + A_j\cos\frac{\lambda_j x}{L_x} + B_j\sinh\frac{\lambda_j x}{L_x} + C_j\cosh\frac{\lambda_j x}{L_x}
\tag{7}
$$

$$V_k(y) = \sin \frac{\lambda_k y}{L_y} + A_k \cos \frac{\lambda_k y}{L_y} + B_k \sinh \frac{\lambda_k y}{L_y} + C_k \cosh \frac{\lambda_k y}{L_y} \tag{8}$$

where A_j, B_j, C_j, A_k, B_k and C_k are constants and λ_j, λ_k are mode frequencies which are determined by using appropriate boundary conditions. Applying the generalized two-dimensional integral transforms (4), equation (2) takes the form

$$
\begin{aligned}
& X_A T(0, L_x, L_y) + X_A F_A^0(t) + \tilde{V}_{tt}(j, k, t) - X_{B1} F_{B1}^0(t) - X_{B2} F_{B2}^0(t) - X_C F_{C1}^0(t) \\
& - X_C F_{C2}^0(t) + X_D \tilde{V}(j, k, t) - X_E F_{B1}^0(t) - X_E F_{B2}^0(t) + F_D^0(t) + F_E^0(t) \\
& + F_F^0(t) + F_G^0(t) = \frac{Mg}{\mu} V_k(y_0) V_j \left(x_0 + ct + \frac{1}{2} at^2\right)
\end{aligned}
\tag{9}
$$

where

$$X_A = \frac{D}{\mu}, \qquad X_{B1} = \frac{N_x}{\mu}, \qquad X_{B2} = \frac{N_y}{\mu}, \qquad X_C = R^0, \qquad X_D = \frac{K}{\mu}, \qquad X_E = \frac{G}{\mu}, \tag{10}$$

$$
\begin{aligned}
T(0, L_x, L_y, t) = & \int_0^{L_y} \left[V_j(x) \frac{\partial^3 V(x,y,t)}{\partial x^3} - V_j'(x) \frac{\partial^2 V(x,y,t)}{\partial x^2} + V_j''(x) \frac{\partial V(x,y,t)}{\partial x} \right. \\
& \left. - V_j''' V(x,y,t) \right]_0^{L_x} V_k(y) dy + \int_0^{L_x} \left[V_k(y) \frac{\partial V(x,y,t)}{\partial y} - V_k'(y) V(x,y,t) \right]_0^{L_y} V_j''(x) dx \\
& + \int_0^{L_y} \left[V_j(x) \frac{\partial V(x,y,t)}{\partial x} - V_j'(x) V(x,y,t) \right]_0^{L_x} V_k''(y) dy + \int_0^{L_x} \left[V_k(y) \frac{\partial^3 V(x,y,t)}{\partial y^3} \right. \\
& \left. - V_k'(y) \frac{\partial^2 V(x,y,t)}{\partial y^2} + V_k''(y) \frac{\partial V(x,y,t)^3}{\partial y} - V_k'''(y) V(x,y,t) \right]_0^{L_y} V_j(x) dx
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
F_A^0(t) = & \int_0^{L_x} \int_0^{L_y} V(x,y,t) V_j^{iv}(x) V_k(y) \sin \frac{k\pi y}{L_y} dx dy \\
& + 2 \int_0^{L_x} \int_0^{L_y} V(x,y,t) V_j^{iv}(x) V_j''(x) V_k''(y) dx dy + \int_0^{L_x} \int_0^{L_y} V(x,y,t) V_j(x) V_k(y) dx dy
\end{aligned}
\tag{12}
$$

$$F_{B1}^0(t) = \int_0^{L_x} \int_0^{L_y} \frac{\partial^2 V(x,y,t)}{\partial x^2} V_j(x) V_k(y) dx dy \tag{13}$$

$$F_{B2}^0(t) = \int_0^{L_x} \int_0^{L_y} \frac{\partial^2 V(x,y,t)}{\partial y^2} V_j(x) V_k(y) dx dy \tag{14}$$

$$F_{C1}^0(t) = \int_0^{L_x} \int_0^{L_y} \frac{\partial^4 V(x,y,t)}{\partial x^2 \partial t^2} V_j(x) V_k(y) dx dy \tag{15}$$

$$F_{C2}^0(t) = \int_0^{L_x} \int_0^{L_y} \frac{\partial^4 V(x,y,t)}{\partial y^2 \partial t^2} V_j(x) V_k(y) dx dy \tag{16}$$

$$F_D^0(t) = \int_0^{L_x} \int_0^{L_y} \frac{M(c+a)^2}{\mu} H\left[x - \left(x_0 + ct + \tfrac{1}{2}at^2\right)\right] H[y - y_0] \frac{\partial^2 V(x,y,t)}{\partial x^2} V_j(x) V_k(y) dx dy \tag{17}$$

$$F_E^0(t) = \int_0^{L_x} \int_0^{L_y} \frac{M}{\mu} H\left[x - \left(x_0 + ct + \tfrac{1}{2}at^2\right)\right] H[y - y_0] \frac{\partial^2 V(x,y,t)}{\partial t^2} V_j(x) V_k(y) dx dy \tag{18}$$

$$F_F^0(t) = \int_0^{L_x} \int_0^{L_y} \frac{2M(c+at)}{\mu} H\left[x - \left(x_0 + ct + \tfrac{1}{2}at^2\right)\right] H[y - y_0] \frac{\partial^2 V(x,y,t)}{\partial x \partial t} V_j(x) V_k(y) dx dy \tag{19}$$

$$F_G^0(t) = \int_0^{L_x} \int_0^{L_y} \frac{aM}{\mu} H\left[x - \left(x_0 + ct + \tfrac{1}{2}at^2\right)\right] H[y - y_0] \frac{\partial V(x,y,t)}{\partial x} V_j(x) V_k(y) dx dy \tag{20}$$

It is recalled that the equation of the free vibration of a rectangular plate is given by

$$D\left[\frac{\partial^4 V(x,y,t)}{\partial x^4} + 2\frac{\partial^4 V(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 V(x,y,t)}{\partial y^4}\right] + \mu \frac{\partial^2 V(x,y,t)}{\partial t^2} = 0 \tag{21}$$

Substituting

$$V(x,y,t) = V_j(x) V_k(y) \cos \omega_{j,k} t \tag{22}$$

into equation (21) above, where $\omega_{j,k}$ is the natural circular frequency of a rectangular plate, we obtain

$$D\left[V_j^{iv}(x)V_k(y) + 2V_j''(x)V_k''(y) + V_j(x)V_k^{iv}(y)\right] - \mu\omega_{j,k}^2 V_j(x)V_k(y) = 0 \tag{23}$$

It is well known that for a simply supported rectangular plate, $\omega_{j,k}^2$ is given by

$$\omega_{j,k}^2 = D_m\left[\frac{j^4\pi^4}{L_x^4} + 2\frac{j^2 k^2 \pi^2}{L_x^2 L_y^2} + \frac{k^4\pi^4}{L_y^4}\right] \tag{24}$$

Multiply equation (23) by $V(x,y,t)$ and integrating with respect to x and y between the limits 0, L_x and 0, L_y respectively, we get

$$\int_0^{L_x} \int_0^{L_y} V(x,y,t) V_j^{iv}(x) V_k(y) dx dy + 2\int_0^{L_x} \int_0^{L_y} V(x,y,t) V_j''(x) V_k''(y) dx dy$$

$$+ \int_0^{L_x} \int_0^{L_y} V(x,y,t) V_j''(x) V_k''(y) dx dy = \frac{\mu}{D} \omega_{j,k}^2 \int_0^{L_x} \int_0^{L_y} V(x,y,t) V_j(x) V_k(y) dx dy \tag{25}$$

In view of (11), we have

$$F_A^0(t) = \frac{\mu}{D}\omega_{j,k}^2 \overline{V}(j,k,t) \tag{26}$$

Since $\overline{V}(p,q,t)$ is just the coefficient of the generalized two-dimensional finite integral transform

$$V(x,y,t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{V_p}\frac{\mu}{V_q}\overline{V}(p,q,t)V_p(x)V_q(y) \tag{27}$$

It follows that

$$V''(x,y,t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{V_p}\frac{\mu}{V_q}\overline{V}(p,q,t)V_p''(x)V_q(y) \tag{28}$$

Therefore integrals (13) – (16) can be rewritten as

$$F_{B1}^0(t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\overline{V}\frac{\mu}{V_p}\frac{\mu}{V_q}(p,q,t)S_{a1}(p,j)S_{a2}(k,q) \tag{29}$$

where

$$S_{a1}(p,j) = \int_0^{L_x}V_p''(x)V_j(x)dx; \qquad S_{a2}(k,q) = \int_0^{L_y}V_k(y)V_q(y)dy \tag{30}$$

$$F_{B2}^0(t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{V_p}\frac{\mu}{V_q}\overline{V}(p,q,t)S_{b1}(p,j)S_{b2}(k,q) \tag{31}$$

and

$$S_{b1}(p,j) = \int_0^{L_x}V_p(x)V_j(x)dx; \qquad S_{b2}(k,q) = \int_0^{L_y}V_k(y)V_q(y)dy \tag{32}$$

$$F_{C1}^0(t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{V_p}\frac{\mu}{V_q}\overline{V}_{tt}(p,q,t)S_{a1}(p,j)S_{a2}(k,q) \tag{33}$$

$$F_{c2}^0(t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{V_p}\frac{\mu}{V_q}\overline{V}_{tt}(p,q,t)S_{b1}(p,j)S_{b2}(k,q) \tag{34}$$

In order to evaluate the integrals (17) – (20), use is made up of Fourier series representation of the Heaviside function namely,

$$H\left[x - (x_0 + ct + \tfrac{1}{2}at^2)\right] = \frac{1}{4} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\left[x - (x_0 + ct + \tfrac{1}{2}at^2)\right]}{2n+1} \tag{35}$$

Similarly

$$H\left[y - y_0\right] = \frac{1}{4} + \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{\sin[(2m+1)(y-y_0)]}{2m+1} \tag{36}$$

Using equations (35) and (36), one obtains

$$F_D^0(t) = \frac{M(c+at)^2}{16\mu} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}(p,q,t) S_{a1}(p,j) S_{a2}(k,q)$$

$$- \frac{M(c+at)^2}{4\pi\mu} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}(p,q,t) \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{c2}(k,q,m) S_{a1}(p,j)$$

$$+ \frac{M(c+at)^2}{4\pi\mu} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}(p,q,t) \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{c1}(k,q,m) S_{a1}(p,j)$$

$$- \frac{M(c+at)^2}{4\pi\mu} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}(p,q,t) \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{d2}(p,j,n) S_{a2}(k,q)$$

$$+ \frac{M(c+at)^2}{4\pi\mu} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}(p,q,t) \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{d1}(p,j,n) S_{a2}(k,q)$$

$$+ \frac{M(c+at)^2}{\mu\pi^2} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}(p,q,t) \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1}\frac{\cos(2m+1)\pi(y-y_0)}{2m+1}\cdot$$
$$S_{d1}(p,j,n) S_{c1}(k,q,m)$$

$$- \frac{M(c+at)^2}{\mu\pi^2} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}(p,q,t) \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1}\frac{\sin(2m+1)\pi(y-y_0)}{2m+1}\cdot$$
$$S_{di}(p,j,n) S_{c2}(k,q,m)$$

$$- \frac{M(c+at)^2}{\mu\pi^2} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}(p,q,t) \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1}\frac{\cos(2m+1)\pi(y-y_0)}{2m+1}\cdot$$
$$S_{d2}(p,j,n) S_{c1}(k,q,m) \tag{37}$$

$$+ \frac{M(c+at)^2}{\mu\pi^2} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}(p,q,t) \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1}\frac{\sin(2m+1)\pi(y-y_0)}{2m+1}\cdot$$
$$S_{d2}(p,j,n) S_{c2}(k,q,m)$$

$$F_E^0(t) = \frac{M}{16\mu} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_{tt}(p,q,t) S_{b1}(p,j) S_{a2}(k,q)$$

$$- \frac{M}{4\mu\pi} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_{tt}(p,q,t) \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{c2}(k,q,m) S_{a2}(p,j)$$

$$+ \frac{M}{4\mu\pi} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_{tt}(p,q,t) \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{c1}(k,q,m) S_{a2}(p,j)$$

$$- \frac{M}{4\mu\pi} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_{tt}(p,q,t) \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{e2}(p,j,n) S_{b1}(k,q)$$

$$+ \frac{M}{4\mu\pi} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_{tt}(p,q,t) \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{e2}(p,j,n) S_{b1}(k,q)$$

$$+ \frac{M}{\mu\pi^2} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_{tt}(p,q,t) \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1}\frac{\cos(2m+1)\pi(y-y_0)}{2m+1}\cdot$$
$$S_{e1}(p,j,n) S_{c3}(k,q,m)$$

$$- \frac{M}{\mu\pi^2} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_{tt}(p,q,t) \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1}\frac{\sin(2m+1)\pi(y-y_0)}{2m+1}\cdot$$
$$S_{e1}(p,j,n) S_{c4}(k,q,m)$$

$$- \frac{M}{\mu\pi^2} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_{tt}(p,q,t) \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1}\frac{\cos(2m+1)\pi(y-y_0)}{2m+1}\cdot$$
$$S_{e2}(p,j,n) S_{c3}(k,q,m) \tag{38}$$

$$+ \frac{M}{\mu\pi^2} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_{tt}(p,q,t) \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1}\frac{\sin(2m+1)\pi(y-y_0)}{2m+1}\cdot$$
$$S_{e2}(p,j,n) S_{c4}(k,q,m)$$

$$F_E^0(t) = \frac{M(c+at)}{8\mu} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_t(p,q,t) S_{a3}(p,j) S_{a2}(k,q)$$

$$- \frac{M(c+at)}{2\mu\pi} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_t(p,q,t) \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{c4}(k,q,m) S_{a3}(p,j)$$

$$+ \frac{M(c+at)}{2\mu\pi} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_t(p,q,t) \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{c3}(k,q,m) S_{a3}(p,j)$$

$$- \frac{M(c+at)}{2\mu\pi} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_t(p,q,t) \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{g2}(p,j,n) S_{a3}(k,q)$$

$$+ \frac{M(c+at)}{2\mu\pi} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_t(p,q,t) \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{g1}(p,j,n) S_{a3}(k,q)$$

$$+ \frac{2M(c+at)}{\mu\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_t(p,q,t) \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} \cdot$$
$$S_{g2}(p,j,n) S_{c3}(k,q,m)$$

$$- \frac{2M(c+at)}{\mu\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_t(p,q,t) \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} \cdot$$
$$S_{g2}(p,j,n) S_{c4}(k,q,m)$$

$$- \frac{2M(c+at)}{\mu\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_t(p,q,t) \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} \cdot$$
$$S_{g2}(p,j,n) S_{c3}(k,q,m)$$

$$+ \frac{2M(c+at)}{\mu\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_t(p,q,t) \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} \cdot$$
$$S_{g2}(p,j,n) S_{c4}(k,q,m) \tag{39}$$

$$F_G^0(t) = \frac{aM}{\mu} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}(p,q,t) S_{a3}(p,j) S_{a2}(k,q)$$

$$- \frac{aM}{4\mu\pi} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}(p,q,t) \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{c4}(k,q,m) S_{a3}(p,j)$$

$$+ \frac{aM}{4\mu\pi} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}(p,q,t) \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{c3}(k,q,m) S_{a3}(p,j)$$

$$- \frac{aM}{4\mu\pi} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} V(p,q,t) \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{g2}(p,j,n) S_{a3}(k,q)$$

$$+ \frac{aM}{4\mu\pi} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}(p,q,t) \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{g1}(p,j,n) S_{a3}(k,q)$$

$$+ \frac{M}{\mu\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} V(p,q,t) \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} \cdot$$
$$S_{g2}(p,j,n) S_{c3}(k,q,m)$$

$$- \frac{aM}{\mu\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} V(p,q,t) \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} \cdot$$
$$S_{g2}(p,j,n) S_{c4}(k,q,m)$$

$$- \frac{aM}{\mu\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_t(p,q,t) \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} \cdot$$
$$S_{g2}(p,j,n) S_{c3}(k,q,m)$$

$$+ \frac{aM}{\mu\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \overline{V}_t(p,q,t) \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} \cdot$$
$$S_{g2}(p,j,n) S_{c4}(k,q,m) \tag{40}$$

$$\widetilde{V}_{tt}(j,k,t) + \omega_{j,k}^2 \widetilde{V}(j,k,t) - \frac{N_x}{\mu} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \widetilde{V}(p,q,t) S_{a1}(p,j) S_{a2}(k,q) - \frac{N_y}{\mu} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \widetilde{V}(p,q,t) S_{b1}(p,j) S_{b2}(k,q)$$

$$+ \frac{K}{\mu} \widetilde{V}(j,k,t) - R_0 \left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \widetilde{V}_{tt}(p,q,t) S_{a1}(p,j) S_{a2}(k,q) + \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \widetilde{V}_{tt}(p,q,t) S_{b1}(p,j) S_{b2}(k,q) \right]$$

$$- \frac{G}{\mu} \left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \widetilde{V}(p,q,t) S_{a1}(p,j) S_{a2}(k,q) + \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \widetilde{V}(p,q,t) S_{b1}(p,j) S_{b2}(k,q) \right]$$

$$+ \Gamma_1 L_x L_y \Bigg\{ (c+at)^2 \Bigg\{ \frac{1}{16} \left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} S_{a1}(p,j) S_{a2}(k,q) \right] - \frac{1}{4\pi} \left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty} \left(\frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{c2}(k,q,m) S_{a1}(p,j) \right. \right.$$

$$- \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{c1}(k,q,m) S_{a1}(p,j) \Bigg) - \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty} \left(\frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{d2}(p,j,n) S_{a2}(k,q) \right.$$

$$\left. \left. - \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{d1}(p,j,n) S_{a2}(k,q) \right) \right]$$

$$+ \frac{1}{\pi^2} \left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{d1}(p,j,n) S_{c1}(k,q,m) \right.$$

$$- \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{di}(p,j,n) S_{c2}(k,q,m)$$

$$- \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{d2}(p,j,n) S_{c1}(k,q,m)$$

$$+ \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{d2}(p,j,n) S_{c2}(k,q,m) \left. \right] \Bigg\} V(p,q,t)$$

$$+ \Bigg\{ \frac{1}{16} \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} S_{b1}(p,j) S_{a2}(k,q) - \frac{1}{\pi} \left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{c2}(k,q,m) S_{a2}(p,j) \right.$$

$$+ \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{c1}(k,q,m) S_{a2}(p,j) - \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{e2}(p,j,n) S_{b1}(k,q)$$

$$+ \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{e2}(p,j,n) S_{b1}(k,q) \left. \right]$$

$$+ \frac{1}{\pi^2} \left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{e1}(p,j,n) S_{c3}(k,q,m) \right.$$

$$- \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{e1}(p,j,n) S_{c4}(k,q,m)$$

$$- \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{e2}(p,j,n) S_{c3}(k,q,m)$$

$$+ \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{e2}(p,j,n) S_{c4}(k,q,m) \left. \right] \Bigg\} \overline{V}_{tt}(p,q,t)$$

$$+ (c + at)\left\{ \frac{1}{8}\left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} S_{a3}(p,j)S_{a2}(k,q) \right] - \frac{1}{2\pi}\left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{c4}(k,q,m)S_{a3}(p,j) \right.\right.$$

$$+ \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{c3}(k,q,m)S_{a3}(p,j)$$

$$- \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{g2}(p,j,n)S_{a3}(k,q)$$

$$+ \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{g1}(p,j,n)S_{a3}(k,q) \Bigg]$$

$$+ \frac{2}{\pi^2}\left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{g2}(p,j,n)S_{c3}(k,q,m) \right.$$

$$- \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{g2}(p,j,n)S_{c4}(k,q,m)$$

$$- \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\mu^2}{V_j V_k} \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{g2}(p,j,n)S_{c3}(k,q,m)$$

$$+ \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{g2}(p,j,n)S_{c4}(k,q,m) \Bigg]\Bigg\}\overline{V}_t(p,q,t)$$

$$+ a\left\{ \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} S_{a3}(p,j)S_{a2}(k,q) - \frac{a}{4\pi}\left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{c4}(k,q,m)S_{a3}(p,j) \right.\right.$$

$$+ \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{c3}(k,q,m)S_{a3}(p,j)$$

$$- \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{g2}(p,j,n)S_{a3}(k,q)$$

$$+ \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} S_{g1}(p,j,n)S_{a3}(k,q) \Bigg]$$

$$+ \frac{1}{\pi^2}\left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{g2}(p,j,n)S_{c3}(k,q,m) \right.$$

$$- \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{g2}(p,j,n)S_{c4}(k,q,m)$$

$$- \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\cos(2m+1)\pi(y-y_0)}{2m+1} S_{g2}(p,j,n)S_{c3}(k,q,m)$$

$$+ \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1} \frac{\sin(2m+1)\pi(y-y_0)}{2m+1} S_{g2}(p,j,n)S_{c4}(k,q,m) \Bigg]\Bigg\}\overline{V}_t(p,q,t)$$

$$= \frac{MgL_x L_y}{\mu\lambda_j\lambda_k}\left[Q_1(\lambda,j) + Cos\frac{\lambda_j}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right) - A_j Sin\frac{\lambda_j}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right) - B_j Cosh\frac{\lambda_j}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right) \right.$$

$$\left. - C_j Sinh\frac{\lambda_j}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right) + Q_2(\lambda,j) - Cos\frac{\lambda_k y_0}{L_y} + A_k Sin\frac{\lambda_k y_0}{L_y} + B_k Cosh\frac{\lambda_k y_0}{L_y} + C_k Sinh\frac{\lambda_k y_0}{L_y} \right] \tag{41}$$

$$Q_1(\lambda,j) = -Cos\,\lambda_j + A_j Sin\,\lambda_j + B_j Cosh\,\lambda_j + C_j Sinh\,\lambda_j \tag{42}$$

$$Q_2(\lambda,k) = -Cos\,\lambda_k + A_k Sin\,\lambda_k + B_k Cosh\,\lambda_k + C_k Sinh\,\lambda_k \tag{43}$$

$$\Gamma_1 = \frac{M}{\mu L_x L_y} \tag{44}$$

$$S_{a1}(p,j) = \int_0^{L_x} V_j(x)V_p''(x))dx,$$

$$S_{a2}(k,q) = \int_0^{L_y} V_k(y)V_q(y)dy \qquad\qquad S_{a3}(p,j) = \int_0^{L_x} V_j(x)V_p'(x)dx$$

$$S_{b1}(p,j) = \int_0^{L_x} V_j(x)V_p(x)dx, \qquad\qquad S_{b2}(k,q) = \int_0^{L_y} V_k(y)V_q''(y)dy$$

$$S_{c1}(k,q,m) = \int_0^{L_y} \sin(2m+1)\pi y\, V_k(y)V_q''(y)dy, \qquad S_{c2}(k,q,m) = \int_0^{L_y} \cos(2m+1)\pi y\, V_k(y)V_q''(y)dy$$

$$S_{c3}(k,q,m) = \int_0^{L_y} \sin(2m+1)\pi y\, V_k(y)V_q(y)dy, \qquad S_{c2}(k,q,m) = \int_0^{L_y} \cos(2m+1)\pi y\, V_k(y)V_q(y)dy$$

$$S_{d1}(p,j) = \int_0^{L_x} \sin(2n+1)\pi x\, V_j(x)V_p''(x)dx, \qquad S_{d2}(p,j) = \int_0^{L_x} \cos(2n+1)\pi x\, V_j(x)V_p''(x)dx$$

$$S_{e1}(p,j) = \int_0^{L_x} \sin(2n+1)\pi x\, V_j(x)V_p(x)dx, \qquad S_{d2}(p,j) = \int_0^{L_x} \cos(2n+1)\pi x\, V_j(x)V_p(x)dx \qquad (45)$$

$$S_{g1}(p,j) = \int_0^{L_x} \sin(2n+1)\pi x\, V_j(x)V_p'(x)dx, \qquad S_{g2}(p,j) = \int_0^{L_x} \cos(2n+1)\pi x\, V_j(x)V_p'(x)dx$$

Equation (41) is the transformed coupled non-homogeneous second order ordinary differential equation describing the transverse motions of the isotropic rectangular plate resting on elastic foundation under the action of partially distributed masses travelling at non-uniform velocity. It is now the fundamental equation of our dynamical problem and holds for all arbitrary boundary conditions. The initial conditions are given by

$$\overline{V}(j,k,0) = 0; \qquad \overline{V}_t(j,k,0) = 0 \qquad\qquad (46)$$

In what follows, two special cases of equation (41) are discussed.

3.1 Solution of the Transformed Equation

3.1.1 <u>Isotropic rectangular plate traversed by moving partially distributed force</u>

The approximate model of the isotropic rectangular plate which assumes the inertia effect of the moving partially distributed mass M as negligible is obtained when the mass ratio Γ_1 is set to zero in equation (41). Thus, setting $\Gamma_1 = 0$ in equation (41), one obtains

$$
\begin{aligned}
&\tilde{V}_{tt}(j,k,t) + \omega_{j,k}^2 \tilde{V}(j,k,t) - \frac{N_x}{\mu}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\tilde{V}(p,q,t)S_{a1}(p,j)S_{a2}(k,q) - \frac{N_y}{\mu}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\tilde{V}(p,q,t)S_{b1}(p,j)S_{b2}(k,q) \\
&+ \frac{K}{\mu}\tilde{V}(j,k,t) - R_0\left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\tilde{V}_{tt}(p,q,t)S_{a1}(p,j)S_{a2}(k,q) + \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\tilde{V}_{tt}(p,q,t)S_{b1}(p,j)S_{b2}(k,q)\right] \\
&- \frac{G}{\mu}\left[\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\tilde{V}(p,q,t)S_{a1}(p,j)S_{a2}(k,q) + \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\tilde{V}(p,q,t)S_{b1}(p,j)S_{b2}(k,q)\right] = \frac{MgL_xL_y}{\mu\lambda_j\lambda_k}\big[Q_1(\lambda,j) \\
&- \cos\frac{\lambda_j}{L_x}\left(x_0 + ct + \tfrac{1}{2}at^2\right) - A_j\sin\frac{\lambda_j}{L_x}\left(x_0 + ct + \tfrac{1}{2}at^2\right) - B_j\cosh\frac{\lambda_j}{L_x}\left(x_0 + ct + \tfrac{1}{2}at^2\right) \\
&- C_j\sinh\frac{\lambda_j}{L_x}\left(x_0 + ct + \tfrac{1}{2}at^2\right) + Q_2(\lambda,k) - \cos\frac{\lambda_k y_0}{L_y} + A_k\sin\frac{\lambda_k y_0}{L_y} + B_k\cosh\frac{\lambda_k y_0}{L_y} + C_k\sinh\frac{\lambda_k y_0}{L_y}\big]
\end{aligned}
\qquad (47)
$$

This represents the classical case of a moving distributed force problem associated with our dynamical system. Evidently, an analytical solution to equation (47) is not feasible. Consequently, we resort to a modification of the asymptotic technique due to Struble. By this technique, we seek the modified frequency corresponding to the frequency of the free system due to the presence of the effect of the rotatory inertia. An equivalent free system operator defined by the modified frequency then replaces equation (47). To this end, equation (47) is rearranged to take the form

$$
\widetilde{V}_{tt}(j,k,t) + \frac{\left(\beta_{mf}^2 \widetilde{V}(j,k,t) - \dfrac{\varepsilon^* L_x L_y}{\mu R^0}\left[(G+N_x)S_{a1}(j,j)S_{a2}(k,k) + (G+N_y)S_{b1}(j,j)S_{b2}(k,k)\right]\right)}{\left(1 - \varepsilon^* L_x L_y[S_{a1}(j,j)S_{a2}(k,k) + S_{b1}(j,j)S_{b2}(k,k)]\right)}\widetilde{V}(j,k,t)
$$

$$
- \frac{\varepsilon^*}{\left(1 - \varepsilon^* L_x L_y[S_{a1}(j,j)S_{a2}(k,k) + S_{b1}(j,j)S_{b2}(k,k)]\right)}\sum_{\substack{p=1 \\ p \neq j}}^{\infty}\sum_{\substack{q=1 \\ q \neq j}}^{\infty}\left\{\left[L_x L_y S_{a1}(j,j)S_{a2}(k,k)\right.\right.
$$

$$
\left.+ L_x L_y S_{b1}(j,j)S_{b2}(k,k)\right]\widetilde{V}_{tt}(p,q,t) + \frac{L_x L_y}{\mu R_0}\left[(G+N_x)S_{a1}(j,j)S_{a2}(k,k) + (G+N_y)S_{b1}(j,j)S_{b2}(k,k)\right]\right\}\widetilde{V}(p,q,t)
$$

$$
= \frac{P^0}{\left(1 - \varepsilon^* L_x L_y[S_{a1}(j,j)S_{a2}(k,k) + S_{b1}(j,j)S_{b2}(k,k)]\right)}\left\{Q_1(\lambda,j) + Cos\frac{\lambda_j}{L_x}\left(x_0 + ct + \tfrac{1}{2}at^2\right) - A_j Sin\frac{\lambda_j}{L_x}\left(x_0 + ct + \tfrac{1}{2}at^2\right)\right.
$$

$$
\left. - B_j Cosh\frac{\lambda_j}{L_x}\left(x_0 + ct + \tfrac{1}{2}at^2\right) - C_j Sinh\frac{\lambda_j}{L_x}\left(x_0 + ct + \tfrac{1}{2}at^2\right)\right]
$$

$$(48)$$

where

$$
\beta_{mf}^2 = \omega_{j,k}^2 + \frac{K}{\mu}, \qquad \varepsilon^* = \frac{R^0}{L_x L_y}, \qquad P^0 = \frac{MgL_x L_y}{\mu\lambda_j\lambda_k}V_k(\lambda_k, y_0) \tag{49}
$$

$$
V_k(\lambda_k, y_0) = Q_2(\lambda,k) - Cos\frac{\lambda_k y_0}{L_y} + A_k Sin\frac{\lambda_k y_0}{L_y} + B_k Cosh\frac{\lambda_k y_0}{L_y} + C_k Sinh\frac{\lambda_k y_0}{L_y} \right] \tag{50}
$$

Thus, we set the right hand side of equation (48) to zero and considering a parameter $\eta^0 < 1$ for any arbitrary ratio ε^* defined as

$$
\eta^0 = \frac{\varepsilon^*}{1 + \varepsilon^*} \tag{51}
$$

It is observed that all the coefficients of the differential operator which acts on $\widetilde{V}(j,k,t)$ and $\widetilde{V}(p,q,,t)$ in equation (48) can be expressed in terms of $O\left(\eta^0\right)$

$$
\varepsilon^* = \eta^0 + O\left(\eta^{0^2}\right) \tag{52}
$$

$$
\frac{1}{\left(1 - \eta^0 L_x L_y[S_{a1}(j,j)S_{a2}(k,k) + S_{b1}(j,j)S_{b2}(k,k)]\right)} = \left(1 - \eta^0 L_x L_y[S_{a1}(j,j)S_{a2}(k,k) + S_{b1}(j,j)S_{b2}(k,k)]\right)^{-1} \text{ where}
$$

$$(53)$$

$$
\left|\eta^0 L_x L_y[S_{a1}(j,j)S_{a2}(k,k) + S_{b1}(j,j)S_{b2}(k,k)]\right| < 1 \tag{54}
$$

Substituting equations (52) and (53) into the homogeneous part of equation (48), one obtains

$$
\widetilde{V}_{tt}(j,k,t) + \left(\beta_{mf}^2 \widetilde{V}(j,k,t) - \frac{\eta^0 L_x L_y}{\mu R_0} \Big[(G + N_x) S_{a1}(j,j) S_{a2}(k,k) + (G + N_y) S_{b1}(j,j) S_{b2}(k,k) \Big] \right) \cdot
$$

$$
\left(1 + \eta^0 L_x L_y \Big[S_{a1}(j,j) S_{a2}(k,k) + S_{b1}(j,j) S_{b2}(k,k) \Big] \right) V(j,k,t) - \eta^0 \Big(1 + \eta^0 L_x L_y \Big) \Big[S_{a1}(j,j) S_{a2}(k,k) \quad (55)
$$

$$
+ S_{b1}(j,j) S_{b2}(k,k) \Big] \sum_{\substack{p=1 \\ p \neq j}}^{\infty} \sum_{\substack{q=1 \\ q \neq j}}^{\infty} \Big\{ \Big[L_x L_y S_{a1}(j,j) S_{a2}(k,k) + L_x L_y S_{b1}(j,j) S_{b2}(k,k) \Big] \widetilde{V}_{tt}(p,q,t)
$$

$$
+ \frac{L_x L_y}{\mu R_0} \Big[(G + N_x) S_{a1}(j,j) S_{a2}(k,k) + (G + N_y) S_{b1}(j,j) S_{b2}(k,k) \Big] \widetilde{V}(p,q,t) \Big\} = 0
$$

when η^0 is set to zero in equation (55) a situation corresponding to the case in which the effect of the cross-sectional dimensions of the rectangular plate is regarded as negligible is obtained,. In such a case, the solution of equation (55) can be obtained as

$$
\widetilde{V}(j,k,t) = A_{mf} Cos\Big[\alpha_{mf} t - \phi_{mf} \Big] \tag{56}
$$

where A_{mf} and ϕ_{mf} are constants.

Since $\eta^0 < 1$, Struble's technique requires that equation (55) be of the form

$$
\widetilde{V}(j,k,t) = C(j,k,t) Cos\Big[\beta_{mf} t - \phi(j,k,t) \Big] + \eta^0 \widetilde{V}_1(j,k,t) + O(\eta^{o^2}) \tag{57}
$$

where $C(j,k,t)$ and $\phi(j,k,t)$ are slowly varying function of time. To obtain the modified frequency, equation (57) and its derivatives are substituted into equation (55). After some simplifications and arrangements, one obtains

$$
\ddot{C}(j,k,t) Cos\Big[\beta_{mf} t - \phi(j,k,t) \Big] + 2\dot{C}(j,k,t)\dot{\phi}(j,k,t) Sin\Big[\beta_{mf} t - \phi(j,k,t) \Big] - 2\dot{C}(j,k,t)\beta_{mf} Sin\Big[\beta_{mf} t - \phi(j,k,t) \Big]
$$

$$
+ C(j,k,t)\ddot{\phi}(j,k,t) Sin\Big[\beta_{mf} t - \phi(j,k,t) \Big] + 2C(j,k,t)\beta_{mf}\dot{\phi}(j,k,t) Cos\Big[\beta_{mf} t - \phi(j,k,t) \Big]
$$

$$
- C(j,k,t)\Big[\dot{\phi}(j,k,t) \Big]^2 Sin\Big[\beta_{mf} t - \phi(j,k,t) \Big] - C(j,k,t)\beta_{mf}^2 Cos\Big[\beta_{mf} t - \phi(j,k,t) \Big] + \eta^0 \ddot{\widetilde{V}}_1(j,k,t) + O(\eta^{o^2})
$$

$$
+ \Big(\beta_{mf}^2 \widetilde{V}(j,k,t) + \eta^0 \beta_{mf}^2 L_x L_y \Big[S_{a1}(j,j) S_{a2}(k,k) + S_{b1}(j,j) S_{b2}(k,k) \Big]
$$

$$
- \frac{\eta^0 L_x L_y}{\mu R^0} \Big[(G + N_x) S_{a1}(j,j) S_{a2}(k,k) + (G + N_y) S_{b1}(j,j) S_{b2}(k,k) \Big] \Big) \Big(C(j,k,t) Cos\Big[\beta_{mf} t - \phi(j,k,t) \Big] \eta^0 \widetilde{V}_1(j,k,t) \Big)
$$

$$
- \eta^0 \sum_{\substack{p=1 \\ p \neq j}}^{\infty} \sum_{\substack{q=1 \\ q \neq j}}^{\infty} \Big[\Big[L_x L_y S_{a1}(j,j) S_{a2}(k,k) + L_x L_y S_{b1}(j,j) S_{b2}(k,k) \Big] \Big(\ddot{C}(p,q,t) Cos\Big[\beta_{mf} t - \phi(p,q,t) \Big]
$$

$$
2\dot{C}(p,q,t)\dot{\phi}(p,q,t) Sin\Big[\beta_{mf} t - \phi(p,q,t) \Big] - 2\dot{C}(p,q,t)\beta_{mf} Sin\Big[\beta_{mf} t - \phi(p,q,t) \Big]
$$

$$
+ C(p,q,t)\ddot{\phi}(p,q,t) Sin\Big[\beta_{mf} t - \phi(p,q,t) \Big] + 2C(p,q,t)\beta_{mf}\dot{\phi}(p,q,t) Cos\Big[\beta_{mf} t - \phi(p,q,t) \Big]
$$

$$
- C(p,q,t)\Big[\dot{\phi}(p,q,t) \Big]^2 Sin\Big[\beta_{mf} t - \phi(p,q,t) \Big] - C(p,q,t)\beta_{mf}^2 Cos\Big[\beta_{mf} t - \phi(p,q,t) \Big] + \eta^0 \ddot{\widetilde{V}}_1(p,q,t) \Big)
$$

$$
+ \frac{L_x L_y}{\mu R^0} \Big[(G + N_x) S_{a1}(j,j) S_{a2}(k,k) + (G + N_y) S_{b1}(j,j) S_{b2}(k,k) \Big] \Big(\ddot{C}(j,k,t) Cos\Big[\beta_{mf} t - \phi(j,k,t) \Big] \eta^0 \widetilde{V}_1(j,k,t) \Big) \Big\} \tag{58}
$$

neglecting terms to $O(\eta^2)$

The variational equations are obtained by equating the coefficients of $Sin\Big[\beta_{mf} t - \phi(j,k,t) \Big]$ and $Cos\Big[\beta_{mf} t - \phi(j,k,t) \Big]$ on both sides of the equation (58) to zero, thus one obtains

$$-2\dot{C}(j,k,t) = 0 \tag{59}$$

and

$$2C(j,k,t)\beta_{mf}\dot{\phi}(j,k,t) + \eta^o\beta_{mf}^2 L_x L_y \left[S_{a1}(j,j)S_{a2}(k,k) + S_{b1}(j,j)S_{b2}(k,k) \right]C(j,k,t)$$

$$-\frac{\eta^0 L_x L_y}{\mu R^0}\left[(G+N_x)S_{a1}(j,j)S_{a2}(k,k) + (G+N_y)S_{b1}(j,j)S_{b2}(k,k) \right]C(j,k,t) = 0 \tag{60}$$

Evaluating equations (59) and (60) one obtains

$$C(j,k,t) = C_{QMF} \tag{61}$$

and

$$\dot{\phi}(j,k,t) = \left\{ \frac{\eta^0 L_x L_y}{2\beta_{mf}\mu R^0}\left[(G+N_x)S_{a1}(j,j)S_{a2}(k,k) + (G+N_y)S_{b1}(j,j)S_{b2}(k,k) \right] \right.$$

$$\left. -\frac{\eta^0 \beta_{mf} L_x L_y}{2}\left[S_{a1}(j,j)S_{a2}(k,k) + S_{b1}(j,j)S_{b2}(k,k) \right] \right\}t + \lambda_{QMF} \tag{62}$$

where C_{QMF} and λ_{QMF} are constants. Therefore when the inertia effect of the plate cross-sectional is considered, the first approximation to the homogeneous system is obtained as

$$\widetilde{V}(j,k,t) = C_{QMF}Cos\left[\sigma_{QMF}t - \lambda_{QMF} \right] \tag{63}$$

Where

$$\sigma_{QMF} = \beta_{mf}\left[1 - \frac{\eta^0}{2\beta_{mf}^2}\left\{ \frac{L_x L_y}{\mu R}\left[(G+N_x)S_{a1}(j,j)S_{a2}(k,k) + (G+N_y)S_{b1}(j,j)S_{b2}(k,k) \right] \right. \right.$$

$$\left. \left. - L_x L_y\left[S_{a1}(j,j)S_{a2}(k,k) + S_{b1}(j,j)S_{b2}(k,k) \right] \right\} \right] \tag{64}$$

represents the modified natural frequency due to the effect of the cross-sectional dimensions of the plate. It is observed that when $\eta^0 = 0$, we recover the frequency of the moving force problem when the effect of the cross-sectional dimension of the plate is neglected. Thus, to solve the non-homogeneous equation (48), the differential operator which act on $\widetilde{V}(j,k,t)$ and $\widetilde{V}(p,q,t)$ is now replaced by the equivalent free system operator defined by the modified frequency σ_{QMF}, i.e

$$\widetilde{V}_{tt}(j,k,t) + \sigma_{QMF}^2\widetilde{V}(j,k,t) = P_{QMF}^0\left[Q_1(\lambda,j) + Cos\frac{\lambda_j}{L_x}\left(x_0 + ct + \tfrac{1}{2}at^2 \right) - A_j Sin\frac{\lambda_j}{L_x}\left(x_0 + ct + \tfrac{1}{2}at^2 \right) \right.$$

$$\left. + B_j Cosh\frac{\lambda_j}{L_x}\left(x_0 + ct + \tfrac{1}{2}at^2 \right) - C_j Sinh\frac{\lambda_j}{L_x}\left(x_0 + ct + \tfrac{1}{2}at^2 \right) \right] \tag{65}$$

where

$$P_{QMF}^0 = \frac{P^0}{\left(1 - \eta^0 L_x L_y\left[S_{a1}(j,j)S_{a2}(k,k) + S_{b1}(j,j)S_{b2}(k,k) \right]\right)} \tag{66}$$

The solution to (65) is obtained using the method variation of parameters in conjunction with Fresnel sine and Fresnel cosine identities and the initial conditions; the solution becomes

$$
\begin{aligned}
\overline{V}(j,k,t) = \frac{P_{QMF}^0}{\sigma_{QMF}} \Big\{ & Sin\,\sigma_{QMF}\,t \big[P_{13}\,S(d_{12}+d_{10}t) + P_{14}\,C(d_{12}+d_{10}t) + P_{11}\,S(d_{11}+d_{10}t) + P_{12}\,C(d_{11}+d_{10}t) \\
& + P_{21}\,C(d_{11}+d_{10}t) - P_{22}\,S(d_{11}+d_{10}t) + P_{23}\,C(d_{12}+d_{10}t) - P_{24}\,S(d_{12}+d_{10}t) - Q_{12}\,erf(d_{21}+d_{20}t) \\
& - Q_{11}\,erfi(d_{21}+d_{20}t) - Q_{13}\,erf(d_{22}+d_{20}t) - Q_{14}\,erfi(d_{22}+d_{20}t) + Q_{21}\,erf(d_{21}+d_{20}t) \\
& - Q_{22}\,erfi(d_{21}+d_{20}t) + Q_{23}\,erf(d_{22}+d_{20}t) - Q_{24}\,erfi(d_{22}+d_{20}t) + \frac{1}{2\sigma_{QMF}}\big[Sin(\lambda_j - \sigma_{QMF}\,t) \\
& - Sin(\lambda_j + \sigma_{QMF}\,t) + A_j\big(Cos(\lambda_j - \sigma_{QMF}\,t) - Cos(\lambda_j + \sigma_{QMF}\,t)\big) - iB_j\big(Sinh(\lambda_j + i\sigma_{QMF}\,t) \\
& - Sinh(\lambda_j - i\sigma_{QMF}\,t)\big) - iC_j\big(Cosh(\lambda_j + i\sigma_{QMF}\,t) - Cosh(\lambda_j - i\sigma_{QMF}\,t)\big)\big] - F_2^* \big] \\
& - Cos\,\sigma_{QMF}\,t \big[-P_{11}\,C(d_{11}+d_{10}t) + P_{12}\,S(d_{11}+d_{10}t) + P_{13}\,C(d_{12}+d_{10}t) - P_{14}\,S(d_{12}+d_{10}t) \\
& + P_{21}\,S(d_{11}+d_{10}t) + P_{22}\,C(d_{11}+d_{10}t) - P_{23}\,S(d_{12}+d_{10}t) - P_{24}\,C(d_{12}+d_{10}t) + iQ_{11}\,erfi(d_{21}+d_{20}t) \\
& - iQ_{12}\,erf(d_{21}+d_{20}t) - iQ_{13}\,erfi(d_{22}+d_{20}t) + iQ_{14}\,erf(d_{22}+d_{20}t) + iQ_{21}\,erf(d_{21}+d_{20}t) \\
& + iQ_{22}\,erfi(d_{21}+d_{20}t) - iQ_{23}\,erf(d_{22}+d_{20}t) - iQ_{24}\,erfi(d_{22}+d_{20}t) \\
& + \frac{1}{2\sigma_{QMF}}\big[Cos(\lambda_j - \sigma_{QMF}\,t) + cos(\lambda_j + \sigma_{QMF}\,t) - A_j\big(Sin(\lambda_j - \sigma_{QMF}\,t) + Sin(\lambda_j + \sigma_{QMF}\,t)\big) \\
& - B_j\big(Cosh(\lambda_j + i\sigma_{QMF}\,t) + Cosh(\lambda_j - i\sigma_{QMF}\,t)\big) - C_j\big(Sinh\,\lambda_j + i\sigma_{QMF}\,t) + Sinh(\lambda_j - i\sigma_{QMF}\,t)\big)\big] + F_1^* \big] \Big\}
\end{aligned}
$$

(67)

Substituting equation (67) into (5), we have

$$
\begin{aligned}
\overline{V}(x,y,t) = \frac{1}{\rho_j(x)\rho_k(y)} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \Big\{ & \frac{P_{QMF}^0\,Sin\,\sigma_{QMF}\,t}{\sigma_{QMF}} \big[P_{13}\,S(d_{12}+d_{10}t) + P_{14}\,C(d_{12}+d_{10}t) + P_{11}\,S(d_{11}+d_{10}t) \\
& + P_{12}\,C(d_{11}+d_{10}t) + P_{21}\,C(d_{11}+d_{10}t) - P_{22}\,S(d_{11}+d_{10}t) + P_{23}\,C(d_{12}+d_{10}t) - P_{24}\,S(d_{12}+d_{10}t) \\
& - Q_{11}\,erfi(d_{21}+d_{20}t) - Q_{12}\,erf(d_{21}+d_{20}t) - Q_{13}\,erf(d_{22}+d_{20}t) - Q_{14}\,erfi(d_{22}+d_{20}t) \\
& + Q_{21}\,erf(d_{21}+d_{20}t) - Q_{22}\,erfi(d_{21}+d_{20}t) + Q_{23}\,erf(d_{22}+d_{20}t) - Q_{24}\,erfi(d_{22}+d_{20}t) \\
& + \frac{1}{2\sigma_{QMF}}\big[Sin(\lambda_j - \sigma_{QMF}\,t) - Sin(\lambda_j + \sigma_{QMF}\,t) + A_j\big(Cos(\lambda_j - \sigma_{QMF}\,t) - Cos(\lambda_j + \sigma_{QMF}\,t)\big) \\
& - iB_j\big(Sinh(\lambda_j + i\sigma_{QMF}\,t) - Sinh(\lambda_j - i\sigma_{QMF}\,t)\big) - iC_j\big(Cosh(\lambda_j + i\sigma_{QMF}\,t) - Cosh(\lambda_j - i\sigma_{QMF}\,t)\big)\big] - F_2^* \big] \\
& - \frac{P_{QMF}^o\,Cos\,\sigma_{QMF}\,t}{\sigma_{QMF}} \big[-P_{11}\,C(d_{11}+d_{10}t) + P_{12}\,S(d_{11}+d_{10}t) + P_{13}\,C(d_{12}+d_{10}t) - P_{14}\,S(d_{12}+d_{10}t) \\
& + P_{21}\,S(d_{11}+d_{10}t) + P_{22}\,C(d_{11}+d_{10}t) - P_{23}\,S(d_{12}+d_{10}t) - P_{24}\,C(d_{12}+d_{10}t) + iQ_{11}\,erfi(d_{21}+d_{20}t) \\
& - iQ_{12}\,erf(d_{21}+d_{20}t) - iQ_{13}\,erfi(d_{22}+d_{20}t) + iQ_{14}\,erf(d_{22}+d_{20}t) + iQ_{21}\,erf(d_{21}+d_{20}t) \\
& + iQ_{22}\,erfi(d_{21}+d_{20}t) - iQ_{23}\,erf(d_{22}+d_{20}t) - iQ_{24}\,erfi(d_{22}+d_{20}t) \\
& + \frac{1}{2\sigma_{QMF}}\big[Cos(\lambda_j - \sigma_{QMF}\,t) + cos(\lambda_j + \sigma_{QMF}\,t) - A_j\big(Sin(\lambda_j - \sigma_{QMF}\,t) + Sin(\lambda_j + \sigma_{QMF}\,t)\big) \\
& + F_1^* \big] V_k(y_0) \Big(Sin\frac{\lambda_j x}{L} + A_j\,Cos\frac{\lambda_j x}{L} + B_j\,Sinh\frac{\lambda_j x}{L} + C_j\,Cosh\frac{\lambda_j x}{L} \Big) \\
& - B_j\big(Cosh(\lambda_j + i\sigma_{QMF}\,t) + Cosh(\lambda_j - i\sigma_{QMF}\,t)\big) - C_j\big(Sinh\,\lambda_j + i\sigma_{QMF}\,t) + Sinh(\lambda_j - i\sigma_{QMF}\,t)\big)\big]
\end{aligned}
$$

(68)

which represents the transverse response to a partially distributed forces, moving at variable velocities of a prestressed isotropic rectangular plate resting on Pasternak elastic foundation and having arbitrary end support conditions.

3.1.2 Isotropic rectangular plate traversed by moving distributed mass

In this section the mass of the moving load is considered commensurable with that of the structure, that is, the inertia effect of the moving mass is not negligible. Thus, the mass ratio $\Gamma_1 \neq 0$ and the solution to the entire equation (41) are required. This gives the moving mass problem. An exact analytical solution to this dynamical problem is not feasible. Thus, as in the previous section, the modified asymptotic technique due to Struble is resorted to. Evidently, the homogeneous part of equation (41) can be replaced by a free system operator defined by the modified frequency due to the presence of rotatory inertia correction factor. To this end, equation (41) can now be rearranged to take the form

$$
\begin{aligned}
&\widetilde{V}_{tt}(j,k,t) + \frac{\Gamma_1 G_B}{1+\Gamma_1 G_A}\widetilde{V}_t(j,k,t) + \frac{\left(\sigma_{QMF}^2 - \Gamma_1 G_C\right)}{1+\Gamma_1 G_A}\widetilde{V}(j,k,t) + \frac{\Gamma_1 G_B}{1+\Gamma_1 G_A}\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq j}}^{\infty}\left\{G_a(p,q,n,m,t)\widetilde{V}_{tt}(j,k,t)\right.\\
&+ G_a(p,q,n,m,t)\widetilde{V}_t(j,k,t)\Big\} = \frac{MgL_x L_y V_k(y_0)}{\mu\lambda_j\lambda_k(1+\Gamma_1 G_A)}\left[Q_1(\lambda,j)+Cos\frac{\lambda_j}{L_x}\left(x_0+ct+\tfrac{1}{2}at^2\right)\right.\\
&- A_j Sin\frac{\lambda_j}{L_x}\left(x_0+ct+\tfrac{1}{2}at^2\right) - B_j Cosh\frac{\lambda_j}{L_x}\left(x_0+ct+\tfrac{1}{2}at^2\right) - C_j Sin\frac{\lambda_j}{L_x}\left(x_0+ct+\tfrac{1}{2}at^2\right)\Bigg]
\end{aligned}
\tag{69}
$$

Where

$$
\begin{aligned}
G_A =&\ \frac{1}{4}S_{b1}(j,j)S_{a2}(k,k) + \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{Cos(2m+1)\pi y_0}{2m+1}S_{c1}(k,k,m)S_{b1}(j,j)\\
&- \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{Sin(2m+1)\pi y_0}{2m+1}S_{c2}(k,k,m)S_{b1}(j,j) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi}{2n+1}\left(x_0+ct+\tfrac{1}{2}at^2\right)S_{e1}(j,j,n)S_{a2}(k,k)\\
&- \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi}{2n+1}\left(x_0+ct+\tfrac{1}{2}at^2\right)S_{e2}(j,j,n)S_{a2}(k,k)\\
&+ \frac{4}{\pi^2}\sum_{n=0}^{\infty}\left(\frac{Cos(2n+1)\pi}{2n+1}\left(x_0+ct+\tfrac{1}{2}at^2\right)S_{d1}(j,j,n) - \frac{Sin(2n+1)\pi}{2n+1}\left(x_0+ct+\tfrac{1}{2}at^2\right)S_{d2}(j,j,n)\right)\cdot\\
&\sum_{m=0}^{\infty}\left(\frac{Cos(2m+1)\pi y_0}{2m+1}S_{c1}(k,k,m) - \frac{Sin(2m+1)\pi y_0}{2m+1}S_{c2}(k,k,m)\right)
\end{aligned}
\tag{70}
$$

$$
\begin{aligned}
G_B =&\ (c+at)\Bigg\{\frac{1}{4}S_{a3}(j,j)S_{a2}(k,k) + \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{Cos(2m+1)\pi y_0}{2m+1}S_{a3}(j,j)S_{c1}(k,k,m)\\
&- \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{Sin(2m+1)\pi y_0}{2m+1}S_{a3}(j,j)S_{c2}(k,k,m) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi}{2n+1}\left(x_0+ct+\tfrac{1}{2}at^2\right)S_{a2}(k,k)S_{g1}(j,j,n)\\
&- \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi}{2n+1}\left(x_0+ct+\tfrac{1}{2}at^2\right)S_{a2}(k,k)S_{g2}(j,j,n)\\
&+ \frac{4}{\pi^2}\sum_{n=0}^{\infty}\left(\frac{Cos(2n+1)\pi}{2n+1}\left(x_0+ct+\tfrac{1}{2}at^2\right)S_{g1}(j,j,n) - \frac{Sin(2n+1)\pi}{2n+1}\left(x_0+ct+\tfrac{1}{2}at^2\right)S_{g2}(j,j,n)\right)\cdot\\
&\sum_{m=0}^{\infty}\left(\frac{Cos(2m+1)\pi y_0}{2m+1}S_{c1}(k,k,m) - \frac{Sin(2m+1)\pi y_0}{2m+1}S_{c2}(k,k,m)\right)\Bigg\}
\end{aligned}
\tag{71}
$$

and

$$G_C = (c+at)^2 \left\{ \frac{1}{4} S_{a1}(j,j) S_{a2}(k,k) + \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Cos(2m+1)\pi y_0}{2m+1} S_{a1}(j,j) S_{c2}(k,k,m) \right.$$

$$-\frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Sin(2m+1)\pi y_0}{2m+1} S_{a1}(j,j) S_{c1}(k,k,m) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{Cos(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{d1}(j,j,n) S_{a2}(k,k)$$

$$-\frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Sin(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{d2}(j,j,n) S_{a2}(k,k)$$

$$+\frac{4}{\pi^2} \sum_{n=0}^{\infty} \left(\frac{Cos(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{d1}(j,j,n) - \frac{Sin(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{d2}(j,j,n) \right) \cdot$$

$$\sum_{m=0}^{\infty} \left(\frac{Cos(2m+1)\pi y_0}{2m+1} S_{c1}(k,k,m) - \frac{Sin(2m+1)\pi y_0}{2m+1} S_{c2}(k,k,m) \right) \right\} + a \left\{ \frac{1}{4} S_{a3}(j,j) S_{a2}(k,k) \right.$$

$$+\frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Cos(2m+1)\pi y_0}{2m+1} S_{a3}(j,j) S_{c1}(k,k,m) - \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Sin(2m+1)\pi y_0}{2m+1} S_{a3}(j,j) S_{c2}(k,k,m)$$

$$+\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{Cos(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{a3}(k,k) S_{g1}(j,j,n) - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{Sin(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) \cdot$$

$$S_{a3}(k,k) S_{g2}(j,j,n) + \frac{4}{\pi^2} \sum_{n=0}^{\infty} \left(\frac{Cos(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{g1}(j,j,n) \right.$$

$$\left. -\frac{Sin(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{g2}(j,j,n) \right) \sum_{m=0}^{\infty} \left(\frac{Cos(2m+1)\pi y_0}{2m+1} S_{c1}(k,k,m) - \frac{Sin(2m+1)\pi y_0}{2m+1} S_{c2}(k,k,m) \right) \right\}$$

(72)

$$G_a(p,q,m,n,t) = \frac{1}{4} S_{b1}(p,j) S_{a2}(k,q) + \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Cos(2m+1)\pi y_0}{2m+1} S_{c1}(k,q,m) S_{b1}(p,j)$$

$$-\frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Sin(2m+1)\pi y_0}{2m+1} S_{c2}(k,q,m) S_{b1}(p,j) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{Cos(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{e1}(p,j,n) S_{a2}(k,q)$$

$$-\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{Sin(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{e2}(p,j,n) S_{a2}(k,q)$$

$$+\frac{4}{\pi^2} \sum_{n=0}^{\infty} \left(\frac{Cos(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{d1}(p,j,n) - \frac{Sin(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{d2}(p,j,n) \right) \cdot$$

$$\sum_{m=0}^{\infty} \left(\frac{Cos(2m+1)\pi y_0}{2m+1} S_{c1}(k,q,m) - \frac{Sin(2m+1)\pi y_0}{2m+1} S_{c2}(k,q,m) \right)$$

(73)

$$G_b(p,q,m,n,t) = (c+at) \left\{ \frac{1}{4} \widetilde{V}_t(p,q,t) S_{a3}(p,j) S_{a2}(k,q) + \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Cos(2m+1)\pi y_0}{2m+1} S_{a3}(p,j) S_{c1}(k,q,m) \right.$$

$$-\frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Sin(2m+1)\pi y_0}{2m+1} S_{a3}(p,j) S_{c2}(k,q,m) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{Cos(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{a2}(k,q) S_{g1}(p,j,n)$$

$$-\frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Sin(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{a2}(k,q) S_{g2}(p,j,n)$$

$$+\frac{4}{\pi^2} \sum_{n=0}^{\infty} \left(\frac{Cos(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{g1}(p,j,n) - \frac{Sin(2n+1)\pi}{2n+1} \left(x_0+ct+\tfrac{1}{2}at^2\right) S_{g2}(p,j,n) \right) \cdot$$

$$\sum_{m=0}^{\infty} \left(\frac{Cos(2m+1)\pi y_0}{2m+1} S_{c1}(k,q,m) - \frac{Sin(2m+1)\pi y_0}{2m+1} S_{c2}(k,q,m) \right) \right\}$$

(74)

$$G_c(p,q,m,n,t) = (c+at)^2 \left\{ \frac{1}{4} S_{a1}(p,j) S_{a2}(k,q) + \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Cos(2m+1)\pi y_0}{2m+1} S_{a1}(p,j) S_{c2}(k,q,m) \right.$$

$$- \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Sin(2m+1)\pi y_0}{2m+1} S_{a1}(p,j) S_{c1}(k,q,m) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{Cos(2n+1)\pi}{2n+1} \left(x_0 + ct + \tfrac{1}{2} at^2\right) S_{d1}(p,j,n) S_{a2}(k,q)$$

$$- \frac{1}{\pi} \sum_{n=0}^{\infty} \tilde{V}(p,q,t) \frac{Sin(2n+1)\pi}{2n+1} \left(x_0 + ct + \tfrac{1}{2} at^2\right) S_{d2}(p,j,n) S_{a2}(k,q)$$

$$+ \frac{4}{\pi^2} \sum_{n=0}^{\infty} \left(\frac{Cos(2n+1)\pi}{2n+1} \left(x_0 + ct + \tfrac{1}{2} at^2\right) S_{d1}(p,j,n) - \frac{Sin(2n+1)\pi}{2n+1} \left(x_0 + ct + \tfrac{1}{2} at^2\right) S_{d2}(p,j,n) \right) \cdot$$

$$\sum_{m=0}^{\infty} \left(\frac{Cos(2m+1)\pi y_0}{2m+1} S_{c1}(k,q,m) - \frac{Sin(2m+1)\pi y_0}{2m+1} S_{c2}(k,q,m) \right) \right\} + a \left\{ \frac{1}{4} S_{a3}(p,j) S_{a2}(k,q) \right.$$

$$+ \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Cos(2m+1)\pi y_0}{2m+1} S_{a3}(p,j) S_{c1}(k,q,m) - \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Sin(2m+1)\pi y_0}{2m+1} S_{a3}(p,j) S_{c2}(k,q,m)$$

$$+ \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{Cos(2n+1)\pi}{2n+1} \left(x_0 + ct + \tfrac{1}{2} at^2\right) S_{a3}(k,q) S_{g1}(p,j,n) - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{Sin(2n+1)\pi}{2n+1} \left(x_0 + ct + \tfrac{1}{2} at^2\right) S_{a3}(k,q) S_{g2}(p,j,n)$$

$$+ \frac{4}{\pi^2} \sum_{n=0}^{\infty} \left(\frac{Cos(2n+1)\pi}{2n+1} \left(x_0 + ct + \tfrac{1}{2} at^2\right) S_{g1}(p,j,n) - \frac{Sin(2n+1)\pi}{2n+1} \left(x_0 + ct + \tfrac{1}{2} at^2\right) S_{g2}(p,j,n) \right) \cdot$$

$$\left. \sum_{m=0}^{\infty} \left(\frac{Cos(2m+1)\pi y_0}{2m+1} S_{c1}(k,q,m) - \frac{Sin(2m+1)\pi y_0}{2m+1} S_{c2}(k,q,m) \right) \right\} \tag{75}$$

Going through similar argument as in the previous section, we obtain the first approximation to the homogeneous system when the effect of the mass of the partially distributed load is considered as

$$\tilde{V}(j,k,t) = D_{MM} e^{-\eta_M \overline{H}_1(j,k,m)t} Cos\left[\delta_{MM} - \phi_{mm}\right] \tag{76}$$

where D_{MM} and ϕ_{mm} are constants.

$$\overline{H}_1(j,k,m) = \frac{c}{8} S_{a3}(j,j) S_{a2}(k,k) + \frac{c}{\pi} \sum_{m=0}^{\infty} \frac{Cos(2m+1)\pi y_0}{2m+1} S_{a3}(j,j) S_{c1}(k,k,m)$$

$$- \frac{c}{\pi} \sum_{m=0}^{\infty} \frac{Sin(2m+1)\pi y_0}{2m+1} S_{a3}(j,j) S_{c2}(k,k,m) \tag{77}$$

$$\eta_M = \frac{\Gamma_1}{1+\Gamma_1} \tag{78}$$

$$\delta_{MM} = \sigma_{QMF} \left\{ 1 - \frac{\eta_M}{2} \left[\overline{H}_2(j,k,m) - \frac{\left[\overline{H}_3(j,k,m) + \overline{H}_4(j,k,m)\right]}{\sigma_{QMF}^2} \right] \right\} \tag{79}$$

is called the modified natural frequency representing the frequency of the free system due to the presence of moving partially distributed mass and

$$\overline{H}_2(j,k,m) = \frac{1}{4} S_{b1}(j,j) S_{a2}(k,k) + \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Cos(2m+1)\pi y_0}{2m+1} S_{b1}(j,j) S_{c1}(k,k,m)$$

$$- \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{Sin(2m+1)\pi y_0}{2m+1} S_{b1}(j,j) S_{c2}(k,k,m) \tag{80}$$

$$\overline{H}_3(j,k,m) = \frac{c^2}{4} S_{a1}(j,j) S_{a2}(k,k) + \frac{c^2}{\pi} \sum_{m=0}^{\infty} \frac{Cos(2m+1)\pi y_0}{2m+1} S_{a1}(j,j) S_{c2}(k,k,m)$$

$$- \frac{c^2}{\pi} \sum_{m=0}^{\infty} \frac{Sin(2m+1)\pi y_0}{2m+1} S_{a1}(j,j) S_{c1}(k,k,m)$$

(81)

$$\overline{H}_4(j,k,m) = \frac{a}{4} S_{a3}(j,j) S_{a2}(k,k) + \frac{a}{\pi} \sum_{m=0}^{\infty} \frac{Cos(2m+1)\pi y_0}{2m+1} S_{a3}(j,j) S_{c1}(k,k,m)$$

$$- \frac{a}{\pi} \sum_{m=0}^{\infty} \frac{Sin(2m+1)\pi y_0}{2m+1} S_{a3}(j,j) S_{c2}(k,k,m)$$

(82)

To solve the non-homogeneous equation (69) the differential operator which act on $\widetilde{V}(j,k,t)$ and $\widetilde{V}(p,q,t)$ is replaced by equivalent free operator defined by the modified frequency δ_{MM} , i.e

$$V_{tt}(j,k,t) + \delta_{MM}^2 \widetilde{V}(j,k,t) = \eta_M L_x L_y g \Big[Q_1(\lambda, j) + Cos \frac{\lambda_j}{L_x} \Big(x_0 + ct + \tfrac{1}{2} at^2 \Big)$$

$$- A_j Sin \frac{\lambda_j}{L_x} \Big(x_0 + ct + \tfrac{1}{2} at^2 \Big) - B_j Cosh \frac{\lambda_j}{L_x} \Big(x_0 + ct + \tfrac{1}{2} at^2 \Big) - C_j Sin \frac{\lambda_j}{L_x} \Big(x_0 + ct + \tfrac{1}{2} at^2 \Big) \Big]$$

(83)

Evidently, equation (83) is analogous to (65). Thus using the same arguments as in the previous sections, solution to equation (83) can be obtained as

$$\overline{V}(x,y,t) = \frac{1}{\rho_j(x)\rho_k(y)} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\eta_M L_x L_y g}{\delta_{MM}} \{ Sin\,\delta_{MM}\,t[P_{13}S(d_{12} + d_{10}t) + P_{14}C(d_{12} + d_{10}t)$$

$$+ P_{11}S(d_{11} + d_{10}t) + P_{12}C(d_{11} + d_{10}t) + P_{21}C(d_{11} + d_{10}t) - P_{22}S(d_{11} + d_{10}t) + P_{23}C(d_{12} + d_{10}t)$$

$$- P_{24}S(d_{12} + d_{10}t) - Q_{12}erf(d_{21} + d_{20}t) - Q_{11}erfi(d_{21} + d_{20}t) - Q_{13}erf(d_{22} + d_{20}t)$$

$$- Q_{14}erfi(d_{22} + d_{20}t) + Q_{21}erf(d_{21} + d_{20}t) - Q_{22}erfi(d_{21} + d_{20}t) + Q_{23}erf(d_{22} + d_{20}t)$$

$$- Q_{24}erfi(d_{22} + d_{20}t) + \frac{1}{2\delta_{MM}} \Big[Sin(\lambda_j - \delta_{MM}t) - Sin(\lambda_j + \delta_{MM}t) + A_j \Big(Cos(\lambda_j - \delta_{MM}t)$$

$$- Cos(\lambda_j + \delta_{MM}t) \Big) - iB_j \Big(Sinh(\lambda_j + \delta_{MM}t) - Sinh(\lambda_j - i\delta_{MM}t) \Big)$$

$$+ P_{12}S(d_{11} + d_{10}t) + P_{13}C(d_{12} + d_{10}t) - P_{14}S(d_{12} + d_{10}t) + P_{21}S(d_{11} + d_{10}t) + P_{22}C(d_{11} + d_{10}t)$$

$$- P_{23}S(d_{12} + d_{10}t) - P_{24}C(d_{12} + d_{10}t) + iQ_{11}erfi(d_{21} + d_{20}t) - iQ_{12}erf(d_{21} + d_{20}t)$$

$$- iQ_{13}erfi(d_{22} + d_{20}t) + iQ_{14}erf(d_{22} + d_{20}t) + iQ_{21}erf(d_{21} + d_{20}t) + iQ_{22}erfi(d_{21} + d_{20}t)$$

$$- iC_j \Big(Cosh(\lambda_j + i\delta_{MM}t) - Cosh(\lambda_j - i\delta_{MM}t) \Big) \Big] - F_2^* \Big] - Cos\,\delta_{MM}t \Big[-P_{11}C(d_{11} + d_{10}t)$$

$$- iQ_{23}erf(d_{22} + d_{20}t) - iQ_{24}erfi(d_{22} + d_{20}t) + \frac{1}{2\delta_{MM}t} \Big[Cos(\lambda_j - \delta_{MM}t) + cos(\lambda_j + \delta_{MM}t)$$

$$- A_j \Big(Sin(\lambda_j - \delta_{MM}t) + Sin(\lambda_j + \delta_{MM}t) \Big) - B_j \Big(Cosh(\lambda_j + i\delta_{MM}t) + Cosh(\lambda_j - i\delta_{MM}t) \Big)$$

(84)

$$- C_j \Big(Sinh\,\lambda_j + i\delta_{MM}t) + Sinh(\lambda_j - i\delta_{MM}t) \Big) \Big] + F_1^* \} V_k(y_0) \cdot$$

$$\Big(Sin \frac{\lambda_j x}{L} + A_j Cos \frac{\lambda_j x}{L} + B_j Sinh \frac{\lambda_j x}{L} + C_j Cosh \frac{\lambda_j x}{L} \Big)$$

Equation (84) represents the transverse displacement response to travelling partially masses moving at non-uniform velocities of an isotropic rectangular plate resting on Vlasov foundation for various end conditions.

4 Illustrative Examples

In this section, practical examples of classical boundary conditions are selected to illustrate the analyses presented in this paper.

4.1 Rectangular Plate Clamped At Edges $x=0, x=L_x$ with Simple Supports at $y=0, y=L_y$

In this example, the rectangular plate is clamped at edges $x = 0$, $x = L_x$ with simple supports at edges $y = 0$, $y = L_y$. The boundary conditions at such opposite edges are

$$V(0, y, t) = 0, \quad V(L_x, y, t) = 0 \quad and \quad V(x, 0, t) = 0, \quad V(x, L_y, t) = 0 \tag{85}$$

$$\frac{\partial^2 V(0, y, t)}{\partial x^2} = 0, \quad \frac{\partial^2 V(L_x, y, t)}{\partial x^2} = 0 \quad and \quad \frac{\partial^2 V(x, 0, t)}{\partial y^2} = 0, \quad \frac{\partial^2 V(x, L_y, t)}{\partial y^2} = 0 \tag{86}$$

and hence for normal modes, one obtains

$$V_j(0) = 0, \quad V_j(L_x) = 0 \quad and \quad V_k(0) = 0, \quad V_k(L_y) = 0 \tag{87}$$

$$\frac{\partial^2 V_j(0)}{\partial x^2} = 0, \quad \frac{\partial^2 V_j(L_x)}{\partial x^2} = 0 \quad and \quad \frac{\partial^2 V_k(0)}{\partial y^2} = 0, \quad \frac{\partial^2 V_k(L_y)}{\partial y^2} = 0 \tag{88}$$

Using the boundary conditions (85) and (86) in (7) and (8), the following values of the constants and frequency equations are obtained for the clamped edges

$$A_j = -\frac{Sinh\ \lambda_j - Sin\ \lambda_j}{Cosh\ \lambda_j - Cos\ \lambda_j} \Rightarrow A_p = -\frac{Sinh\ \lambda_p - Sin\ \lambda_p}{Cosh\ \lambda_p - Cos\ \lambda_p} \tag{89}$$

$$B_j = -1 \quad \Rightarrow \quad B_p \quad and \quad C_j = -A_j \quad \Rightarrow \quad C_p = -A_p \tag{90}$$

The frequency equation of the clamped edges is given by

$$Cosh\lambda_j Cos\lambda_j - 1 = 0 \tag{91}$$

such that

$$\lambda_1 = 4.73004, \quad \lambda_2 = 7.85320, \quad \lambda_3 = 10.99561 \tag{92}$$

$$\begin{aligned}
V_j = \frac{\mu L_x L_y}{2} \Bigg\{ &1 + A_j^2 - B_j^2 + C_j^2 + \frac{1}{\lambda_j}\Big[2C_j - 2A_j B_j - B_j C_j - \frac{1}{2}\left(1 - A_j^2\right)Sin\lambda_j + 2A_j Sin^2\lambda_j \\
&+ \left(B_j^2 + C_j^2\right)Sinh\lambda_j Cosh\lambda_j + 2\left(B_j - A_j C_j\right)Cosh\lambda_j Sinh\lambda_j + 2\left(-B_j + A_j C_j\right)Sinh\lambda_j Cosh\lambda_j \\
&+ 2\left(C_j - A_j B_j\right)Sinh\lambda_j Sin\lambda_j + 2\left(-C_j + A_j B_j\right)Cosh\lambda_j Cos\lambda_j + B_j C_j Cosh\lambda_j \Big]\Bigg\}
\end{aligned} \tag{93}$$

V_p is obtained by replacing subscripts j by p in equation (95). For the simple edges, it is readily shown that

$$A_k(0) = 0, \quad V_j(L_x) = 0 \quad and \quad V_k(0) = 0, \quad V_k(L_y) = 0 \tag{94}$$

The corresponding frequency equation yields

$$\lambda_k = k\pi \Rightarrow \lambda_q = q\pi \tag{95}$$

And

$$V_k = \frac{\mu L_y}{2}, \quad V_q = \frac{\mu L_y}{2} \tag{96}$$

Thus, the general solutions of the associated moving partially distributed force and moving partially distributed mass problems of the simple-clamped rectangular plate are obtained by substituting the above results in (89) to (96) into equations (68) and (84).

4.2 Rectangular Plate Clamped At All Edges

For the isotropic rectangular plate clamped at all edges, both deflection and the slope vanish at such ends. Thus, the following boundary conditions pertains

$$V(0, y, t) = 0, \quad V(L_x, y, t) = 0 \quad and \quad V(x, 0, t) = 0, \quad V(x, L_y, t) = 0 \tag{97}$$

$$\frac{\partial V(0, y, t)}{\partial x} = 0, \quad \frac{\partial V(L_x, y, t)}{\partial x} = 0 \quad and \quad \frac{\partial V(x, 0, t)}{\partial y} = 0, \quad \frac{\partial V(x, L_y, t)}{\partial y} = 0 \tag{98}$$

and hence for the normal modes, we have

$$V_j(0) = 0, \quad V_j(L_x) = 0 \quad and \quad V_k(0) = 0, \quad V_k(L_y) = 0 \tag{99}$$

$$\frac{\partial V_j(0)}{\partial x} = 0, \quad \frac{\partial V_j(L_x)}{\partial x} = 0 \quad and \quad \frac{\partial V_k(0)}{\partial y} = 0, \quad \frac{\partial V_k(L_y)}{\partial y} = 0 \tag{100}$$

Using the boundary conditions (97) to (100) in equations (7) and (8) one obtains the following values of the constants for the clamped edges at $x = 0,\ x = L_x$

$$A_j = -\frac{Sinh\lambda_j - Sin\lambda_j}{Cosh\lambda_j - Cos\lambda_j} \Rightarrow A_p = -\frac{Sinh\lambda_p - Sin\lambda_p}{Cosh\lambda_p - Cos\lambda_p} \tag{101}$$

$$B_j = -1 \Rightarrow B_p = -1, \quad C_j = -A_j \Rightarrow C_p = -A_p \tag{102}$$

The frequency equation of the clamped edges is given by

$$Cosh\lambda_j Cos\lambda_j - 1 = 0 \tag{103}$$

such that

$$\lambda_1 = 1.875, \quad \lambda_2 = 4.694, \quad \lambda_3 = 7.855 \tag{104}$$

V_j is defined by (93) and V_p is obtained by replacing subscripts j by p

Similarly, for the clamped edges $y = 0, \ y = L_y$ the same process is followed to obtain

$$A_k = -\frac{Sinh\lambda_k - Sin\lambda_k}{Cosh\lambda_k - Cos\lambda_k} \Rightarrow A_q = -\frac{Sinh\lambda_q - Sin\lambda_q}{Cosh\lambda_q - Cos\lambda_q} \tag{105}$$

The frequency equation of the clamped edges is given by

$$Cosh\lambda_k Cos\lambda_k - 1 = 0 \tag{106}$$

such that

$$\lambda_1 = 1.875, \quad \lambda_2 = 4.694, \quad \lambda_3 = 7.855 \tag{107}$$

V_k and V_q are obtained by replacing subscripts j with k and q in equation (93) respectively. Thus, the general solutions of the associated moving partially distributed force and moving partially distributed mass problems of the clamped-clamped rectangular plate are obtained by substituting the above results in (101) to (107) into equations (68) and (84).

5 Comments on Closed form Solutions

At this juncture, in an undamped system such as this, it is pertinent to establish conditions under which resonance occurs. Resonance takes place when the motion of the vibrating structure becomes unbounded. That is, when the vibrations become intensive and can cause catastrophic failure in improperly constructed structures including bridges, buildings and airplanes. The resonance conditions for the boundary conditions are now established and It is evident from equation (68) that the isotropic rectangular plate resting on a Pasternak elastic foundation and traversed by partially distributed forces moving with variable velocities reaches a state of resonance whenever

$$\sigma_{QMF} = \frac{\lambda_j v_c}{L_x}, \qquad \sigma_{QMF} = \frac{\lambda_j v_c}{L_x} + \frac{2at_0}{L_x} \tag{108}$$

while equation (84) shows that the same plate under the action of moving mass experiences a state of resonance whenever

$$\delta_{MM} = \frac{\lambda_j v_c}{L_x}, \ \delta_{MM} = \frac{\lambda_j v_c}{L_x} + \frac{2at_0}{L_x} \tag{109}$$

From the expression (79),

$$\delta_{MM} = \sigma_{QMF}\left\{1 - \frac{\eta_M}{2}\left[\overline{H}_2\left(j,k,m\right) - \frac{\left[\overline{H}_3\left(j,k,m\right) + \overline{H}_4\left(j,k,m\right)\right]}{\sigma_{QMF}^2}\right]\right\} \tag{110}$$

which implies

$$\sigma_{QMF} = \frac{\dfrac{\lambda_j v_c}{L_x}}{1 - \dfrac{\eta_M}{2}\left[\overline{H}_2\left(j,k,m\right) - \dfrac{\left[\overline{H}_3\left(j,k,m\right) + \overline{H}_4\left(j,k,m\right)\right]}{\sigma_{QMF}^2}\right]} \tag{111}$$

It is therefore evident from (109) and (111) that for the same frequency, the critical velocity for the system consisting of isotropic rectangular plate resting on a Pasternak foundation and traversed by partially distributed forces moving at non-uniform velocities is greater than that of the moving mass problem. Thus, for the same natural frequency of an isotropic rectangular plate, resonance is reached earlier in moving mass system than in moving force system.

6 Numerical Results and Discussion

In this section, results are presented and discussed for an isotropic rectangular plate of lengths, $E = 2.109 \times 10^9 \, N/m^2$ a moment of inertia $I = 2.87698 \times 10^{-3} \, m^4$, the plate thickness $h = 0.35$, and the Poisson ratios $v = 0.55$ is considered. The velocity of the travelling partially distributed load is $30 \, m/s$, the value of bending rigidity D is 10000, mass per unit length $\mu = 2758.291 kg/m$. Furthermore, the values of foundation stiffness K is varied between $0 \, N/m^3$ and $400000 \, N/m^3$, the values of axial forces N_x and N_y varied between $0 \, N$ and $2.0 \times 10^8 \, N$, the shear modulus G is varied between $0 \, N/m$ and $3.0 \times 10^7 \, N/m$.

Fig. 6.1 shows the deflection profile of a clamped-clamped isotropic rectangular plate under the action of partially distributed forces moving at variable velocity for various values of foundation stiffness K and fixed values of axial force $N_x = 20000$, shear modulus $G = 10000$ and rotatory inertia correction factor $R^0 = 0.5$ is displayed. The figure shows that as the value of K increases, the transverse displacement of the isotropic rectangular plate decreases. Similar results are obtained when the clamped-clamped plate is subjected to partially distributed masses as shown in Fig. 6.5. For various travelling time t, the transverse displacements of the plate for various values of axial force N_x and fixed values of foundation stiffness $K = 40000$, shear modulus $G = 10000$ and rotatory inertia correction factor $R^0 = 0.5$ are shown in Fig. 6.2. It is observed that higher values of axial force N_x reduce the transverse displacement of the plate. The same behaviour characterises the deflection profile of the clamped-clamped plate under the action of partially distributed masses moving at variable velocity for various values of axial force N_x as shown in Fig. 6.6. Also, Fig. 6.3 displays the response amplitudes of the clamped-clamped isotropic rectangular plate respectively to partially distributed forces travelling at variable velocity for various values of shear modulus G and fixed values of foundation stiffness $K = 40000$, axial force $N_x = 20000$ and rotatory inertia correction factor $R^0 = 0.5$. It is seen from the figure that as the value of the shear modulus increases, the response amplitude of the simply supported isotropic rectangular plate under the action of partially distributed forces travelling at variable velocity decreases. Similar results are obtained when the clamped-clamped isotropic rectangular plate is subjected to a partially distributed masses travelling at variable velocity as shown in Fig. 6.7. In Fig. 6.4 the deflection profile of clamped-clamped isotropic rectangular plate under the action of partially distributed forces is displayed. It is clearly shown that as we increase the values of a rotatory inertia correction factor R^0 for fixed values of foundation stiffness K, axial force N_x

and shear modulus G the deflection of the isotropic plate decreases. Also, for the same clamped-clamped plate traversed by non-uniform partially distributed masses in Fig. 6.8 depicts that as the values of rotatory inertia R^0 increase the deflection of the plate reduces for fixed values of foundation stiffness $K = 40000$, axial force $N_x = 20000$ and shear modulus $G = 10000$.
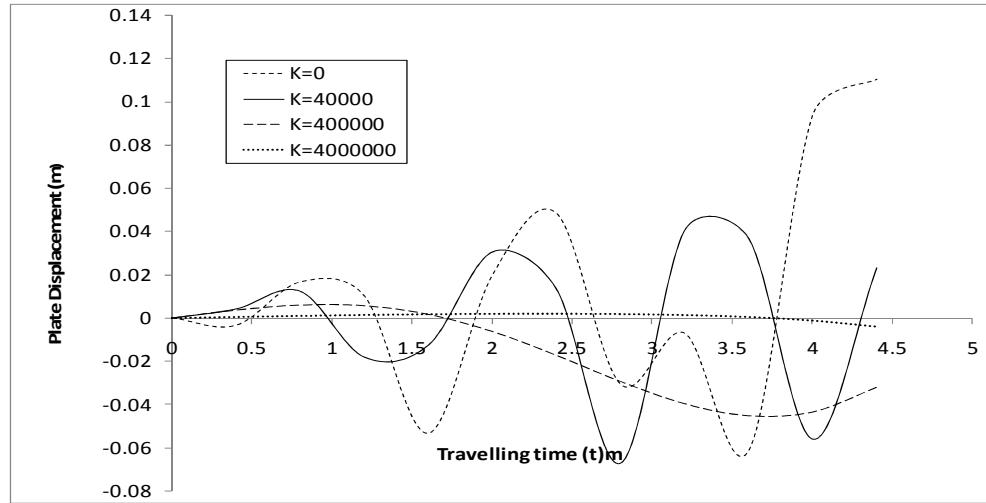

Fig. 6.1. Transverse displacement of a clamped-clamped rectangular plate under partially distributed forces for various values of K and $N_x = 20000$, $G = 10000$, $R^0 = 0.5$.
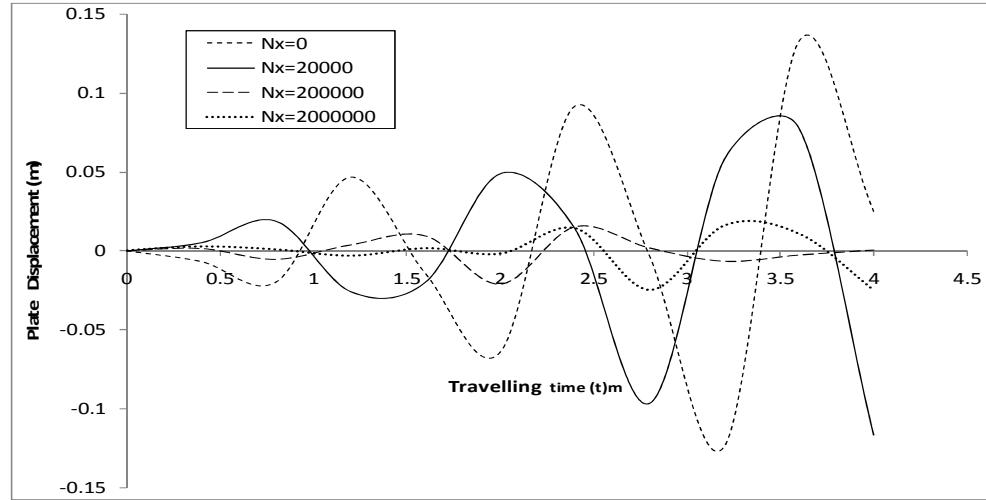
Fig. 6.2. Deflection profile of a clamped-clamped isotropic rectangular plate under partially distributed forces for various values of N and $K = 40000$, $G = 10000$ $R^0 = 0.5$.
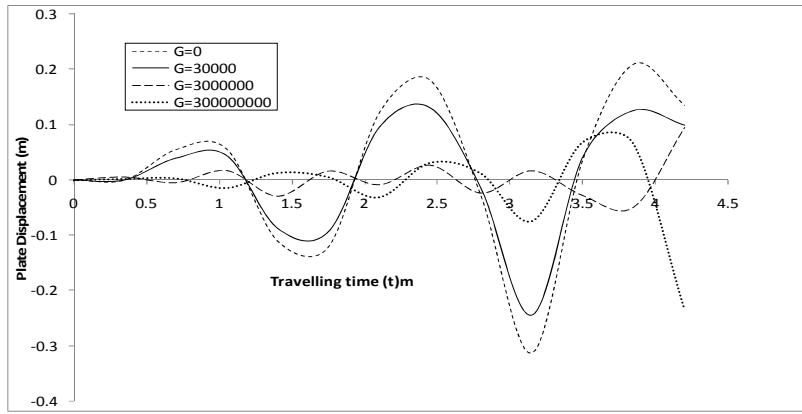
Fig. 6.3. Transverse displacement of a clamped-clamped isotropic rectangular plate under partially distributed forces for various values of G and $K = 40000$, $N_x = 20000$, $R^0 = 0.5$.
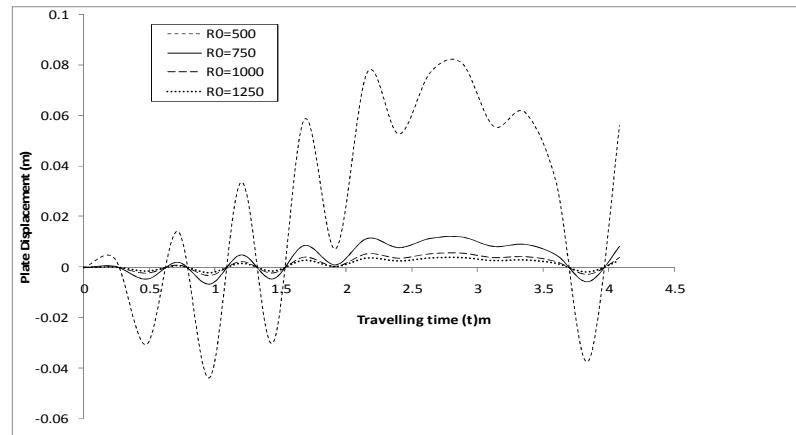

Fig. 6.4. Response amplitude of a clamped-clamped isotropic rectangular plate under partially distributed forces for various values of $\mathbf{R^0}$ and fixed values of $K = 40000$, $N_x = 20000$ **and** $G = 10000$.
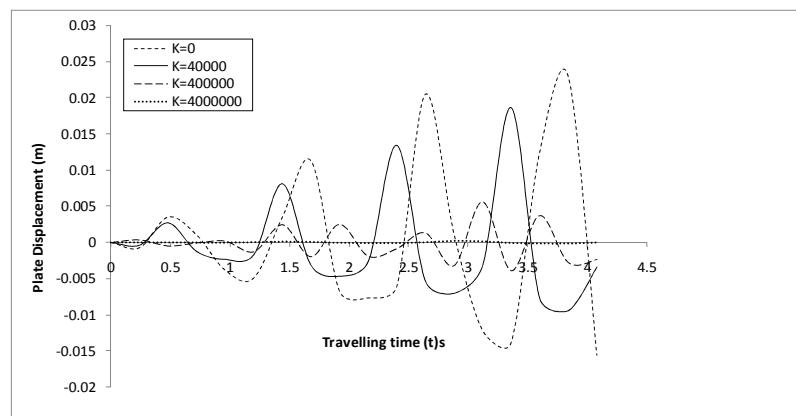
Fig. 6.5. Displacement response of a clamped-clamped isotropic rectangular plate under partially distributed masses for various values of foundation stiffness K and $N_x = 20000$, $G = 10000$, $R^0 = 0.5$.
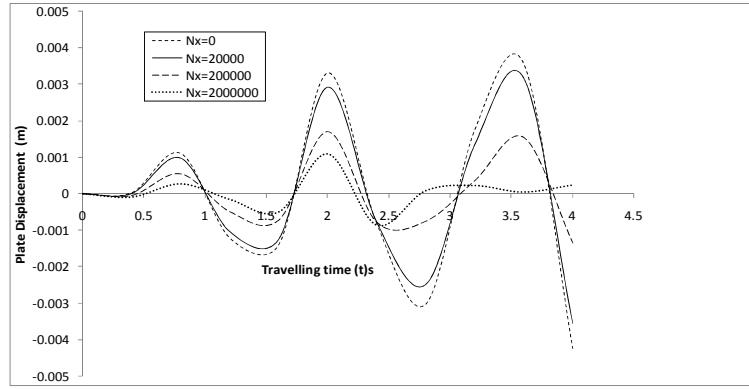
Fig. 6.6. Deflection profile of a clamped-clamped isotropic rectangular plate under partially distributed masses for various values of N and $K = 40000$, $G = 10000$, $R^0 = 0.5$.
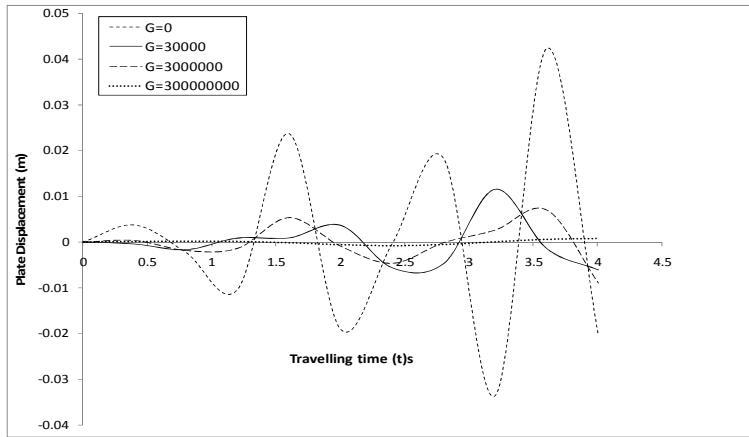

Fig. 6.7. Transverse displacement of a clamped-clamped isotropic rectangular plate under partially distributed for various values of G and $K = 40000$, $N_x = 20000$, $R^0 = 0.5$.
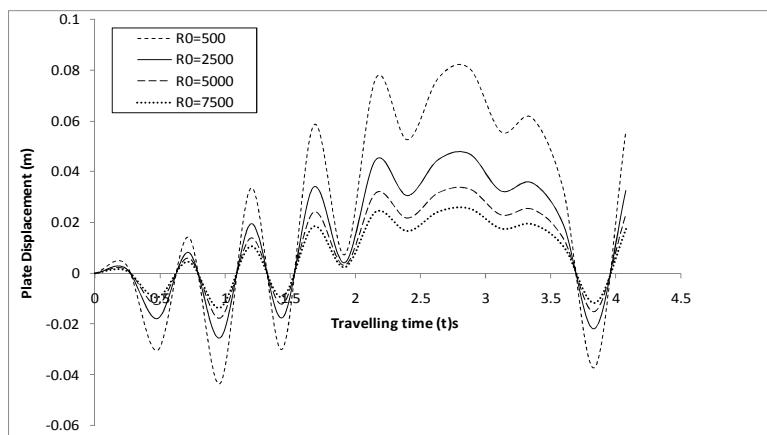
**Fig. 6.8. Response amplitude of a clamped-clamped isotropic rectangular plate under partially distributed masses for various values of R^0 and $K = 40000$, $N_x = 20000$, $G = 10000$.**
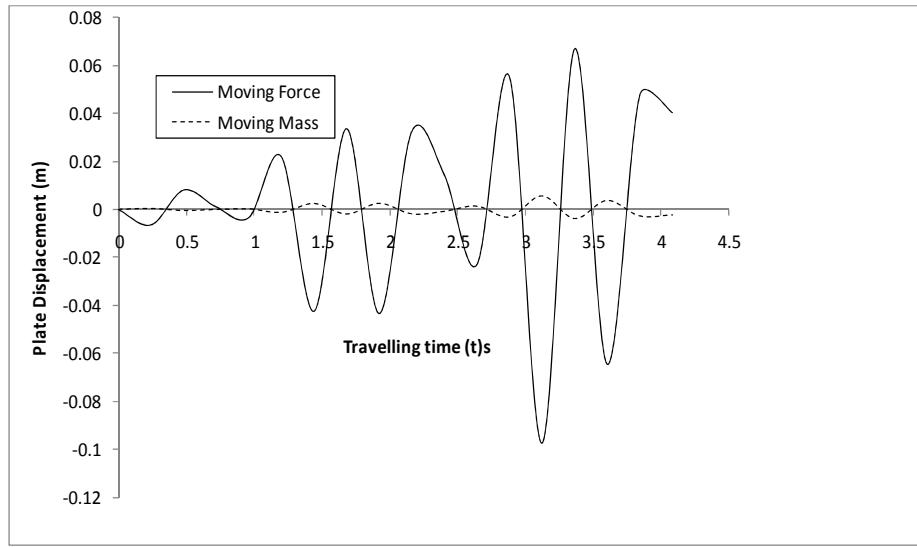
Fig. 6.9. Comparison of the displacement response of moving force and moving mass cases for a non-uniform clamped-clamped plate for fixed values of $K = 400000,$ $N_x = 20000$ $G = 100000$ and $R^0 = 0.5$

Fig. 6.9 displays the comparison of the transverse displacement response of moving force and moving mass cases of the clamped-clamped isotropic rectangular plate traversed by a moving load travelling at variable velocity for fixed values of $K = 400000$, $N_x = 200000$, $G = 100000$ and $R^0 = 0.5$. In Fig. 6.10, the deflection profile of a simple-clamped isotropic rectangular plate under the action of partially distributed forces moving at variable velocity for various values of foundation stiffness K and for fixed values of axial force $N_x = 20000$, shear modulus $G = 10000$ and rotatory inertia correction factor $R^0 = 0.5$ is displayed. The figure shows that as K increases, the deflection of the plate decreases.
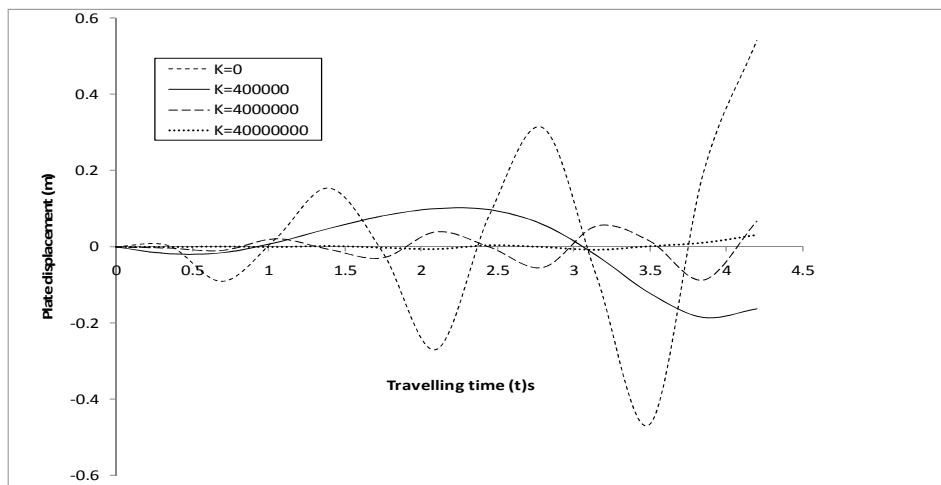
Fig. 6.10. Transverse displacement of a simple-clamped isotropic rectangular plate under partially distributed forces for various values of K and $N_x = 20000$, $G = 10000,$ $R^0 = 0.5$.
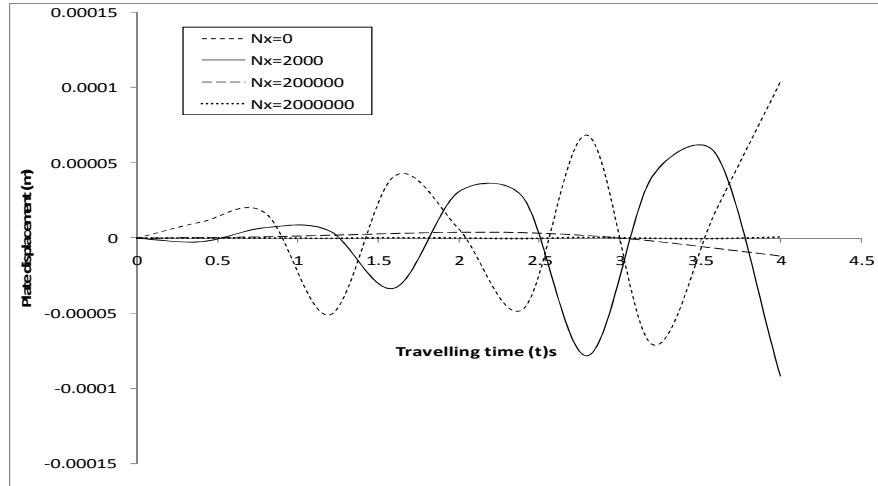
Fig. 6.11. Deflection profile of a simple-clamped isotropic rectangular plate under partially distributed for various values of axial force and $K = 40000$, $G = 10000$, $R^0 = 0.5$ **.**
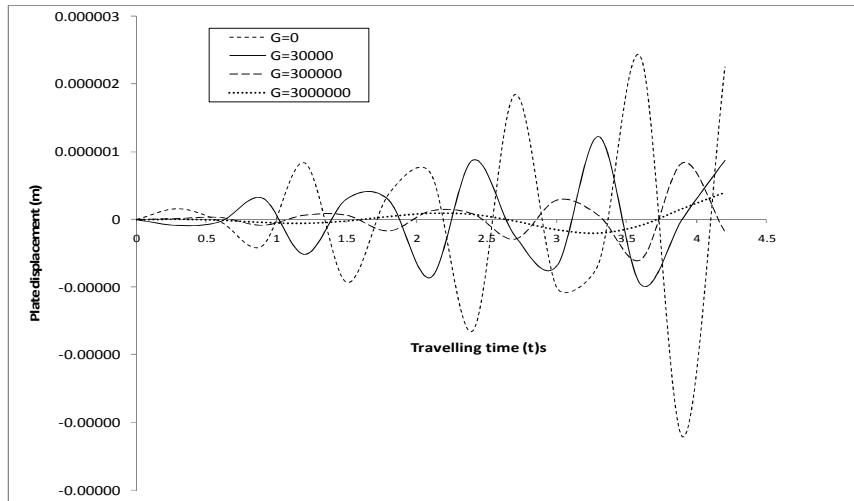

Fig. 6.12. Transverse displacement of a simple-clamped isotropic rectangular plate forces under partially distributed forces for various values of G and $K = 40000$, $N_x = 20000$, $R^0 = 0.5$ **.**

Similar results are obtained when the simple-clamped isotropic rectangular plate is subjected to a partially distributed masses travelling at variable velocity as shown in Fig. 6.14. For various travelling time t, the deflection profiles of the plate for various values of axial force N_x and for fixed values of foundation stiffness K=40000, shear modulus G=10000 and rotatory inertia correction factor $R^0 = 0.5$ are shown in Fig. 6.11. It is observed that higher values of axial force N_x reduce the deflection profile of the plate. The same behaviour characterizes the deflection profile of the simple-clamped isotropic rectangular plate under the action of partially distributed masses moving at variable velocity for various values of axial force N_x as shown in Fig. 6.15. Also, Fig. 6.12 displays the displacement response of the simple-clamped isotropic rectangular plate to partially distributed forces travelling at variable velocity for various values of shear modulus G and for fixed values of foundation stiffness K=40000, axial force $N_x = 20000$ and rotatory

inertia correction factor $R^0 = 0.5$. The same results are obtained when the simple-clamped isotropic rectangular plate is subjected to a partially distributed masses travelling at variable velocity as shown in Fig. 6.16. The transverse displacement response of a simple-clamped isotropic rectangular plate under the action of partially distributed forces moving at variable velocity for various values of rotatory inertia correction factor R^0 and fixed values of foundation stiffness K=40000, axial force $N_x = 20000$ and shear modulus G=10000 is displayed in Fig. 6.13. The figure shows that as R^0 increases, the dynamic deflection of the plate decreases. Similar results are obtained when the simple-clamped isotropic rectangular plate is subjected to a partially distributed masses travelling at variable velocity as shown in Fig 6.17. Finally, Fig. 6.18 depicts the comparison of the transverse displacement response of moving force and moving a mass of a simple-clamped isotropic rectangular plate traversed by a moving load travelling at variable velocity for fixed values of K=400000, $N_x = 200000$, G=100000 and $R^0 = 0.5$
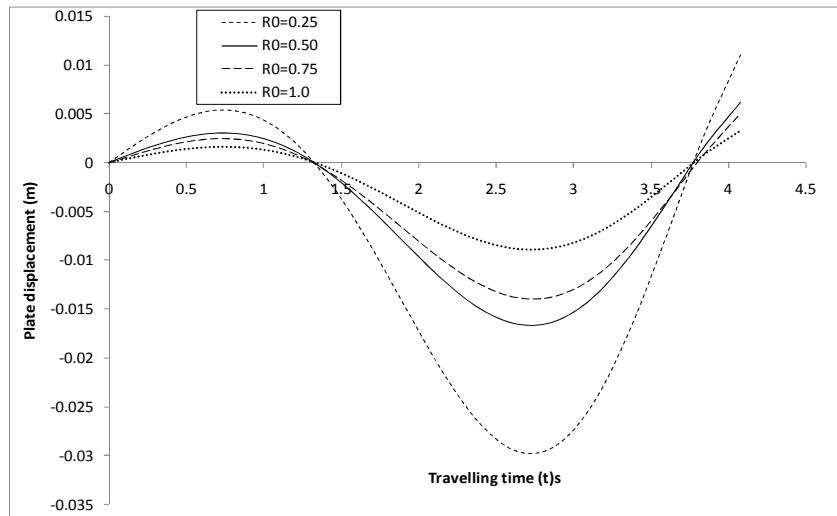

Fig. 6.13. Response amplitude of a simple-clamped isotropic rectangular plate under partially distributed forces for various values of R^0 and $K = 40000, N_x = 20000, G = 10000$.
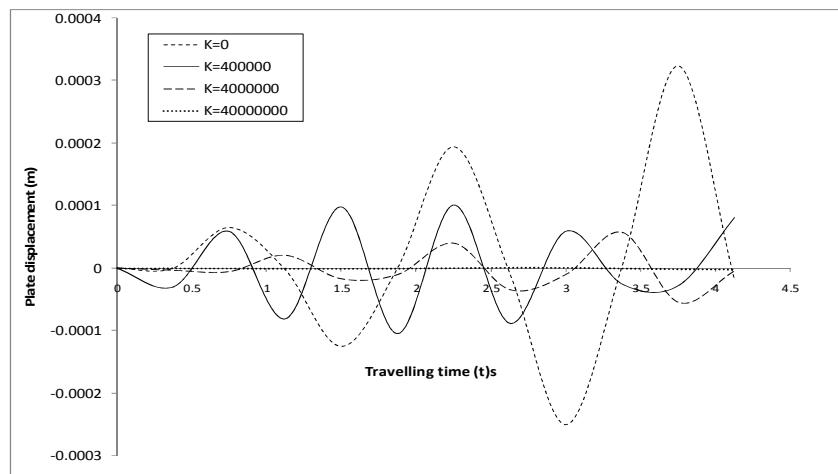

Fig. 6.14. Displacement response of a simple-clamped isotropic rectangular plate under partially distributed masses for various values of K and $N_x = 20000, G = 10000, R^0 = 0.5$.
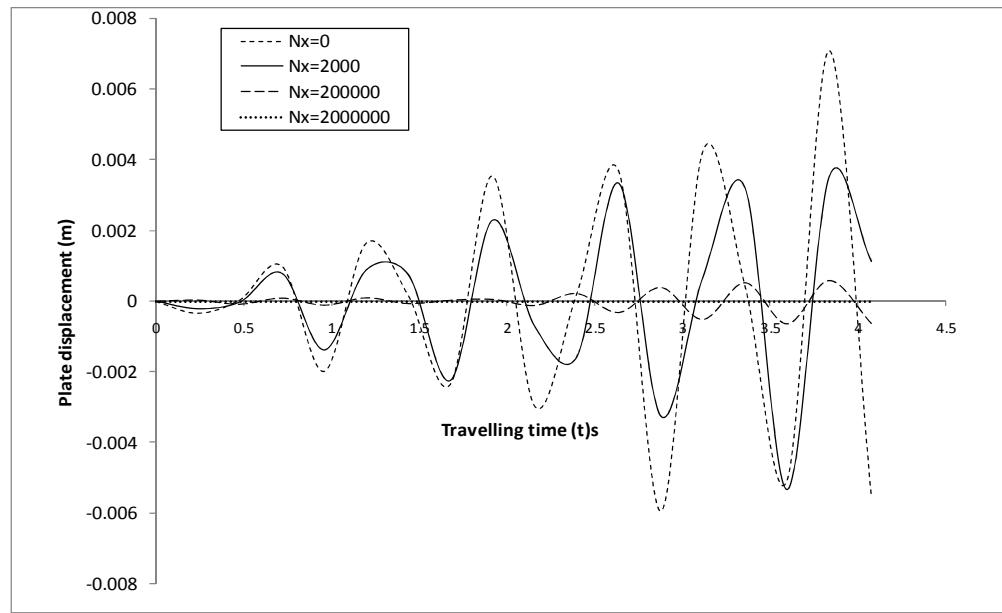
Fig. 6.15. Deflection profile of a simple-clamped isotropic rectangular plate under partially distributed masses for various values of N and $K = 40000,\ G = 10000,\ R^0 = 0.5$ **.**
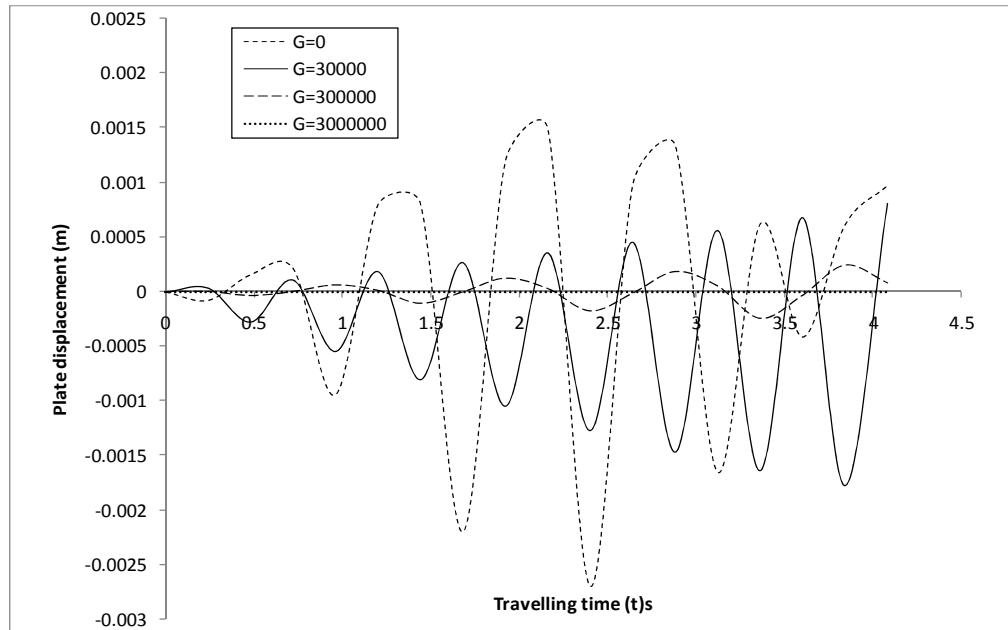

Fig. 6.16. Transverse displacement of a simple-clamped isotropic rectangular plate under partially distributed masses under partially distributed masses for various values of G and $K = 40000,$
$N_x = 20000,\ R^0 = 0.5$ **.**
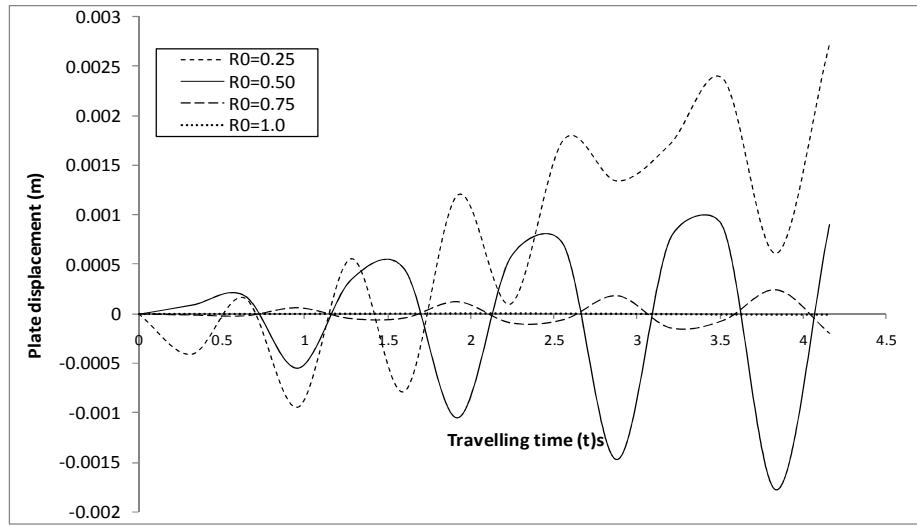
Fig. 6.17. Response amplitude of a simple-clamped isotropic rectangular plate under partially distributed masses for various values of R^0 and $K = 40000$, $N_x = 20000$, $G = 10000$.
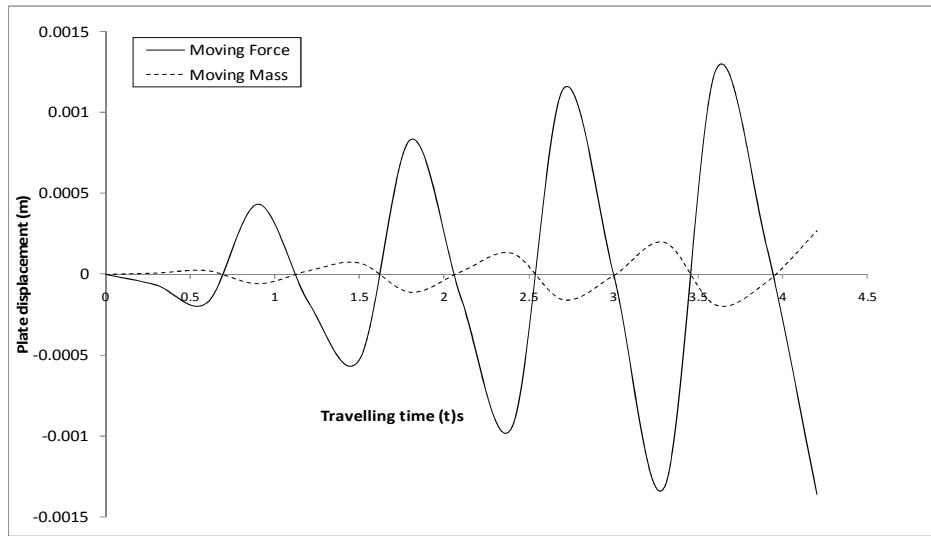

Fig. 6.18. Comparison of the displacement response of moving force and moving mass cases for simple-clamped rectangular plate for fixed values of $K = 400000$, $N_x = 200000$, $G = 100000$ and $R^0 = 0.5$

7 Conclusion

In this paper, the dynamic response of a rectangular plate having arbitrary supports at both ends is presented. The solution technique suitable for all variants of classical boundary conditions involves using the generalised two-dimensional integral transform to reduce the fourth order partial differential equation governing the vibration of the plate to a second ordinary differential equation which is then treated with the modified asymptotic method of Struble. The closed form solutions obtained are analysed and numerical

analyses in plotted curves were presented. The analyses carried out show that as the values of foundation stiffness K, axial force N, shear modulus G and rotatory inertia correction R^o increase, the response amplitude of the plate decrease for all illustrative examples considered. It is shown further from the results that, for the same natural frequency, the critical speed for the for the system traversed by partially distributed moving masses at varying velocities is greater than that of the partially distributed moving force problem for both clamped-clamped and simple-clamped end conditions. Hence, resonance is reached in moving force problem. It is seen that for both end conditions under consideration, the response amplitude of the partially distributed moving force system is higher than that of the partially distributed moving mass system for fixed values of the structural parameters. Thus, increase in the values rotatory inertial correction factor, foundation stiffness and shear modulus reduce the risk of resonance in a vibrating system.

Competing Interests

Authors have declared that no competing interests exist.

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