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Eigenvalues for Some Complex Infinite Tridiagonal Matrices

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

The discrete spectrum for an unbounded operator J defined by a special infinite tridiagonal complex matrix is approximated by the eigenvalues of its orthogonal truncations. Let $\sigma(J)$ means the spectrum of the operator J and

$$\Lambda(J) = \{\lambda \in \mathrm{Lim}_{n\to\infty}\lambda_n : \ \lambda_n \text{ is an eigenvalue of } J_n\},$$

where $\mathrm{Lim}_{n\to\infty}\lambda_n$ is the set of limit points of the sequence (λ_n), and the $n \times n$ matrix J_n is an orthogonal truncation of J.

We consider classes of tridiagonal complex matrices for which $\sigma(J) = \Lambda(J)$.

Keywords: Tridiagonal matrix, complex Jacobi matrix, discrete spectrum, eigenvalue, asymptotic formula, unbounded operator.

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1 Introduction

Examples of infinite non-symmetric tridiagonal matrices and complex Jacobi matrices were investigated by several authors: [1], [2], [3], [4], [5], [6], Ikebe, Asai, Miyazaki and Cai [7], and others (see, e.g., [8], [9], [10], [11], [12], [13], [14] and, [15]). However, systematic investigations, concerning spectral properties of non-selfadjoint tridiagonal operators, are difficult because the structure of complex sequences could be more complicated then the structure of real sequences. Moreover, standard operator methods, which give effective results on real Jacobi matrices, fail in the complex case. The relationships of tridiagonal matrices and the formal orthogonal polynomials on the complex plane can be find, e.g., in [2], [3], [8]. Complex tridiagonal infinite matrices are also useful in the theory of special functions like the Bessel functions or the Mathieu functions [7], [15], and [16]. Non-symmetric tridiagonal matrices with real entries are essentially associated with the Hill equation [4], the Ince equation [15], the Mathieu equation [16], and other second order equations [7], [17].

We consider a complex tridiagonal infinite matrix

$$J\left((d_n),(a_n),(b_n)\right) = \begin{pmatrix} d_1 & a_1 & 0 & \cdots & \cdots \\ b_1 & d_2 & a_2 & 0 & \ddots \\ 0 & b_2 & d_3 & a_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.1}$$

where $d_n,\ a_n,\ b_n \in \mathbb{C}\setminus\{0\}$.

The matrix $J\left((d_n),(a_n),(b_n)\right)$ defines a linear operator J in the space $l_2 = l_2(\mathbb{N})$ which acts on a maximal domain

$$Dom(J) = \{(f_n)_{n=1}^{\infty} \in l_2 :\ (b_{n-1}f_{n-1} + d_n f_n + a_n f_{n+1})_{n=1}^{\infty} \in l_2\} \tag{1.2}$$

and

$$(Jf)_n = b_{n-1}f_{n-1} + d_n f_n + a_n f_{n+1}, \quad n \geq 1, \tag{1.3}$$

for $f = (f_n)_{n=1}^{\infty} \in Dom(J)$ and $b_0 = 0$. The domain $Dom(J)$ of the operator J is dense in l_2.

Assume that X is a complex Banach space. Denote by $B(X)$ the space of all bounded linear operators on X. Let $A : Dom(A) \to X$ be a densely defined linear operator in X. The resolvent set of A is the set

$$\rho(A) = \{\lambda \in \mathbb{C} :\ (A - \lambda I)^{-1} \text{ exists in } B(X)\}. \tag{1.4}$$

The inverse operator $(A - \lambda I)^{-1}$, where $\lambda \in \rho(A)$, is called the resolvent of A and

$$\sigma(A) = \mathbb{C} \setminus \rho(A) \tag{1.5}$$

means the spectrum. We are interested in the classes of unbounded tridiagonal operators, for which the spectrum is discrete. The criteria, which guarantee compactness of the resolvent and a discrete spectrum, for a non-symmetric or complex tridiagonal operator are given in [7], [10], [13] and [15].

Different kind of problems in infinite-dimensional spaces are approximated by some problems in finite-dimensional spaces because of the fact that many mathematical software products provide functionality for the solution of matrix equations. To realize this strategy correctly, the projective and iterative methods can be used, see for example [18], [19], [20], [11] and [21], and the famed monographs: [22], [23], [24], [25]. The question about approximation of the spectrum of an operator by eigenvalues of properly selected matrices is also unsophisticated.

Denote the orthogonal truncation of $J((d_n),(a_n),(b_n))$ by

$$J_n = \begin{pmatrix} d_1 & a_1 & & \\ b_1 & d_2 & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & b_{n-1} & d_n \end{pmatrix} \quad (1.6)$$

for $n \geq 1$.

Define

$$\Lambda(J) = \{\lambda \in \mathrm{Lim}_{n\to\infty}\lambda_n : \ \lambda_n \text{ is an eigenvalue of } J_n\}, \quad (1.7)$$

where $\mathrm{Lim}_{n\to\infty}\lambda_n$ is the set of limit points of the sequence (λ_n).

The aim of the research is finding classes of tridiagonal operators for which $\sigma(J) \subset \Lambda(J)$ or $\sigma(J) \supset \Lambda(J)$. This can be done if we assume that J is an unbounded operator with compact resolvent.

2 $\sigma(J) \subset \Lambda(J)$

Remark 2.1. In the case of self-adjoint operators, among other, the following results are known.

1. If A is a bounded self-adjoint operator in l_2 then $\sigma(A) \subset \Lambda(A)$ [26].

2. If J is a self-adjoint operator given by a real Jacobi matrix, then $\sigma(J) \subset \Lambda(J)$ (classical result for real Jacobi matrices).

The problem of approximation of eigenvalues for unbounded self-adjoint Jacobi matrices acting in l_2 by the eigenvalues of finite matrices was considered also in [27], [28],[29], [30], [31], [17] and in others.

The following results are known for complex or non-selfadjoint tridiagonal operators.

Remark 2.2. If J is represented by a complex tridiagonal matrix $J((d_n),(a_n),(b_n))$ and it is a compact operator, then $\sigma(J) \subset \Lambda(J)$ (classical result, Theorem 18.1, [23]).

Remark 2.3. Real tridiagonal infinite matrices where investigated by H. Volkmer. The enough conditions on real sequences $(a_n),(b_n),(d_n)$ to obtain the inclusion $\sigma(J) \subset \Lambda(J)$ are presented in [15].

Remark 2.4. Y. Ikebe, N. Asai, Y. Miyazaki and D. Cai have proved that if $\lim_{n\to\infty}|d_n| = \infty$ and $(a_n) = (b_n)$ are bounded complex sequences then $\sigma(J) \subset \Lambda(J)$ [7].

We generalize the result given in [7]. The new result for complex tridiagonal operators is as follow.

Theorem 2.1. *If J has a compact resolvent,*

$$\lim_{n\to\infty}|d_n| = \infty$$

and

$$\lim_{n\to\infty}\frac{|a_{n-1}|+|a_n|+|b_{n-1}|+|b_n|}{|d_n|} = 0,$$

then $\sigma(J)$ is discrete and $\sigma(J) \subset \Lambda(J)$.

Proof. The idea of the proof from [7] works in this case. Let C be a diagonal operator in l_2 given by the matrix

$$J\left(\left(\frac{1}{\sqrt{d_n}}\right),(0),(0)\right). \tag{2.1}$$

Obviously, C is compact.

Next, suppose that L is an operator in l_2 defined by

$$J\left((0),(a_n),(b_n)\right). \tag{2.2}$$

Observe that the matrix

$$J\left((0),\left(\frac{a_n}{\sqrt{d_n}\sqrt{d_{n+1}}}\right),\left(\frac{b_n}{\sqrt{d_n}\sqrt{d_{n+1}}}\right)\right) \tag{2.3}$$

represents a compact operator in l_2 if and only if

$$\lim_{n\to\infty}\frac{|a_n|+|b_n|}{\sqrt{|d_n||d_{n+1}|}}=0, \tag{2.4}$$

but the last equation is true because of the assumption. Moreover, CLC has the matrix representation (2.3). Then we can write

$$J\supseteq L+C^{-2}=C^{-1}(CLC+I)C^{-1}. \tag{2.5}$$

Without loss of generality we can assume that J is invertible in $B(l_2)$. Thus $CLC+I$ is also invertible in $B(l_2)$, so

$$A:=J^{-1}=C(CLC+I)^{-1}C \tag{2.6}$$

is a compact operator. Then $Jx=\lambda x$ if and only if $Ax=\frac{1}{\lambda}x$ for $\lambda\in\mathbb{C}$ and $x\in l_2\setminus\{0\}$.

Let $\{e_n : n=1,2,...\}$ be a canonical basis in l_2 and denote by P_n the ortogonal projection on $H_n=\text{span}\{e_1,...,e_n\}$. Then P_n admits the block matrix representation

$$P_n=\begin{pmatrix} I_n & 0\\ 0 & 0\end{pmatrix}, \tag{2.7}$$

where I_n is the identity on H_n.

Denote

$$A_n=P_nC(P_nCLCP_n+I)^{-1}CP_n \tag{2.8}$$

and observe that

$$P_nCLCP_n+I=\begin{pmatrix} C_nJ_nC_n & 0\\ 0 & I_n^{'}\end{pmatrix}, \tag{2.9}$$

where

$$C_n=\begin{pmatrix} \frac{1}{\sqrt{d_1}} & 0 & & \\ 0 & \frac{1}{\sqrt{d_2}} & \ddots & \\ & \ddots & \ddots & 0\\ & & 0 & \frac{1}{\sqrt{d_n}}\end{pmatrix}$$

and $I_n^{'}$ is the identity on $l_2\ominus H_n$.

The operator $K=CLC$ is compact; therefore,

$$\|P_nKP_n-K\|\to 0,\ n\to\infty, \tag{2.10}$$

and also

$$\|(P_nKP_n + I) - (K + I)\| \to 0,\ n \to \infty. \tag{2.11}$$

Notice that $K + I$ is invertible in $B(l_2)$, so there exists n_0 such that $P_nKP_n + I$ is invertible for $n \geq n_0$ and

$$\|(P_nKP_n + I)^{-1} - (K + I)^{-1}\| \to 0,\ n \to \infty. \tag{2.12}$$

Thus A_n exists for $n \geq n_0$; moreover,

$$\|A_n - A\| \leq \tag{2.13}$$
$$\leq \|P_nAP_n - A\| + \|C\|^2\|(P_nKP_n + I)^{-1} - (K + I)^{-1}\|$$

and

$$\|A_n - A\| \to 0,\ n \to \infty. \tag{2.14}$$

The equation (2.8) implies that

$$A_n = \begin{pmatrix} J_n^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.15}$$

So if $\lambda \neq 0$ is an eigenvalue of J then $\mu = \frac{1}{\lambda}$ is an eigenvalue of A. According to the projective method approach ([23], Th. 18.1) we deduce that there exists a sequence $(\mu_n)_{n\geq n_0}$ such that μ_n is an eigenvalue of A_n and $\mu = \lim_{n\to\infty} \mu_n$. Notice that if $\mu_n \neq 0$ is an eigenvalue of A_n then $\lambda_n = \frac{1}{\mu_n}$ is an eigenvalue of J_n. Then $\lambda = \lim_{n\to\infty} \lambda_n$ and finally $\sigma(J) \subset \Lambda(J)$. □

3 $\Lambda(J) \subset \sigma(J)$

The investigations related to the inclusion $\Lambda(J) \subset \sigma(J)$ in the class of tridiagonal infinite matrices are complicated. If $J((d_n), (a_n), (a_n))$ is real and symmetric, then the inclusion $\Lambda(J) \subset \sigma(J)$ holds under some conditions (see [26], [28], [29], [32] and others). The complex case looks interesting too. We successfully use the approach presented in [32], applied to real symmetric Jacobi matrices, for complex tridiagonal matrices.

Theorem 3.1. *Let $\{e_n : n = 1, 2, 3, ...\}$ be the canonical basis for l_2 and P_n be the orthogonal projection on $H_n = \text{span}\{e_1, ..., e_n\}$. Assume that the matrix $J((d_n), (a_n), (b_n))$ defines a linear operator J in l_2 and J_n is given by (1.6). If for all bounded complex sequences $(\lambda_n)_{n=1}^{\infty}$ such that $\lambda_n \in \sigma(J_n)$, $n \geq 1$, and for all sequences $(x_n)_{n=1}^{\infty}$ of eigenvectors such that $P_nJx_n = \lambda_n x_n$, $\|x_n\| = 1$ and $x_n \in H_n$ for $n \geq 1$,*

$$\lim_{n\to\infty} |b_n(x_n, e_n)| = 0, \tag{3.1}$$

then $\Lambda(J) \subset \sigma(J)$.

Proof. We follow [32] and [29]. Let $\lambda \in \Lambda(J)$. Without loss of generality we can assume $\lambda = \lim_{n\to\infty} \lambda_n$, where λ_n is an eigenvalue of J_n. Let $\xi_n = (\xi_{n,1}, \xi_{n,2}, ..., \xi_{n,n})^\top$ be an eigenvector of J_n such that $J_n\xi_n = \lambda_n\xi_n$ and $\|\xi_n\| = 1$ for $n \geq 1$. Then

$$P_nJx_n = P_nJP_nx_n = \lambda_n x_n,$$

where $x_n = \sum_{k=1}^{n} \xi_{n,k}e_k$.

Next notice that

$$Jx_n = \lambda_n x_n + b_n(x_n, e_n)e_{n+1}, \tag{3.2}$$

and

$$\|Jx_n\|^2 = \|\lambda_n\|^2 + \|b_n(x_n, e_n)\|^2. \tag{3.3}$$

Then from

$$\|(J-\lambda)x_n\|^2 = \|Jx_n\|^2 + |\lambda|^2 - 2\mathrm{Re}\overline{\lambda}(Jx_n, x_n), \tag{3.4}$$

(3.2) and (3.3) we derive

$$\|(J-\lambda)x_n\|^2 = |\lambda_n|^2 + |\lambda|^2 - 2\mathrm{Re}\overline{\lambda}\lambda_n + |b_n(x_n, e_n)|^2 = \tag{3.5}$$

$$= |\lambda_n - \lambda|^2 + |b_n(x_n, e_n)|^2. \tag{3.6}$$

So, there exists $(x_n)_{n=1}^{\infty}$ such that $\|x_n\| = 1$ for $n \geq 1$ and $\|(J-\lambda)x_n\| \to 0$, as $n \to \infty$, if (3.1) holds. This ensures that λ belongs to the spectrum of J. □

Now we are going to analyze the condition (3.1). Let (x_n) be a sequence considered in Theorem 3.1, so $x_n = \sum_{k=1}^{n} \xi_{n,k} e_k$ for $n \geq 1$ and $\xi_n = (\xi_{n,1}, \xi_{n,2}, ..., \xi_{n,n})^\top \in \mathbb{C}^n$ is an eigenvector for J_n such that $J_n\xi_n = \lambda_n\xi_n$ and $\|\xi_n\| = \|x_n\| = 1$. Then (3.1) could be writen as

$$\lim_{n\to\infty} |b_n \xi_{n,n}| = 0. \tag{3.7}$$

Next we follow [32] to estimate $|b_n\xi_{n,n}|$ for large n. The equality $J_n\xi_n = \lambda_n\xi_n$ is equivalent to the system

$$\begin{cases} b_{n-1}\xi_{n,n-1} + (d_n - \lambda_n)\xi_{n,n} = 0, \\ b_{n-k-1}\xi_{n,n-k-1} + (d_{n-k} - \lambda_n)\xi_{n,n-k} + a_{n-k}\xi_{n,n-k+1} = 0, \\ \qquad k = 1, \ldots, n-2; \\ (d_1 - \lambda_n)\xi_{n,1} + a_1\xi_{n,2} = 0. \end{cases} \tag{3.8}$$

We have assumed that the sequence (λ_n) is bounded, so $|\lambda_n| \leq M$ for $n \geq 1$. Let $K \geq 1$ be an integer. There exists n_0 such that $|d_{n-K}| > M + 1$ for $n \geq n_0$. Then we use (3.8) to obtain the following estimates

$$|\xi_{n,n-k}| \leq (M+1)\left(\frac{|b_{n-k-1}|}{|d_{n-k}|}|\xi_{n,n-k-1}| + \frac{|a_{n-k}|}{|d_{n-k}|}|\xi_{n,n-k+1}|\right) \tag{3.9}$$

for $k = 1, ..., K$, and

$$|\xi_{n,n}| \leq (M+1)\frac{|b_{n-1}|}{|d_n|}|\xi_{n,n-1}|, \tag{3.10}$$

where $n > n_0$.

Denote

$$B_n = \max\{|a_{n-1}|, ..., |a_{n-K}|, |b_{n-1}|, ..., |b_{n-K}|\} \tag{3.11}$$

and

$$D_n = \min\{|d_n|, ..., |d_{n-K}|\}. \tag{3.12}$$

Follow the idea that was used in [32], from (3.9), (3.10), (3.11) and (3.12) we derive

$$|\xi_{n,n}| \leq c(M,K)\left(\frac{B_n}{D_n}\right)^K \tag{3.13}$$

for large n, where a constant $c(M,K) > 0$ is independent on n.

So if

$$\lim_{n\to\infty} |b_n|\left(\frac{B_n}{D_n}\right)^K = 0, \tag{3.14}$$

then (3.1) holds.

Corollary 3.2. *If J is an operator given by a tridiagonal matrix (1.1), where* $\lim_{n\to\infty}|d_n| = +\infty$, *and there exists an integer $K \geq 1$ such that*

$$\lim_{n\to\infty}|b_n|\left(\frac{B_n}{D_n}\right)^K = 0,$$

where B_n, D_n are given by (3.11) and (3.12), then $\Lambda(J) \subset \sigma(J)$.

Corollary 3.3. *Let J be an operator given by a tridiagonal matrix (1.1) and $|a_n|, |b_n| = O(n^\beta)$ and $\frac{1}{|d_n|} = O(n^{-\alpha})$, as $n \to \infty$. If $\alpha > \beta$ and $\alpha \geq 0$ then $\Lambda(J) \subset \sigma(J)$.*

Proof. If $\alpha > \beta$ then there exists an integer $K \geq 1$ such that $K\alpha > (K+1)\beta$. So

$$|b_n|\left(\frac{B_n}{D_n}\right)^K = O\left(n^{(K+1)\beta - K\alpha}\right), \quad n \to \infty$$

and (3.14) holds. □

4 Conclusion

The assumptions concerning the sequences, that determine the matrix $J((d_n),(a_n),(b_n))$, are very close in Theorem 2.1 as well as in Corollary 3.2. Moreover, we could observe that the existing criteria (see e.g. [7], [10], [13] and [15]), which guarantee compactness of the resolvent for non-symmetric tridiagonal operator are given in the terms of the entries of $J((d_n),(a_n),(b_n))$. So there is large class of non-symmetric tridiagonal matrices such that if an operator J is represented in l_2 by the considered matrix then $\sigma(J) = \Lambda(J)$.

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Competing Interests

Author has declared that no competing interests exist.

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