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Results on Uniqueness of Periodic Solution, Exponential Stability and Controllability Properties of Neutral Delay System

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Authors' contributions

This work was carried out in collaboration between the two authors. Author DKI wrote the protocol and supervises the work. Authors DKI and JA carried out all the research analyses. Author DKI managed the literature searches and edited the manuscript. Both the authors read and approved the final manuscript.

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Abstract

The variation of constant method is employed to evaluate the periodic solution of a linear neutral system with an input function. Uniqueness of the obtained solution is established and proved by utilizing the inversion theory on a perturbed differential operator. The exponential stability of the system equation and the computation of the maximum delay bound for the system to be asymptotically stable are analyzed using the resolvent matrix of the system equation. The controllability of the system is studied by the analyses of the linear ordinary control and the free control parts of the linear neutral system for properness, non-singularity of the gramian matrix, canonical form of the controllable matrix and the non zero/ pole cancellation of the transfer function matrix. Results obtained are employed on neutral delay model of a partial element equivalent circuit (PEEC) consisting of a retarded mutual coupling between the partial inductance to confirm the suitability of the test.

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1 Introduction

The qualitative properties of differential equation models in system theory are greatly improved by the introduction of the delay (neutral) differential equation. This is because the point-wise (instantaneous) reaction of the system for any perturbation is addressed. This important attribute has endeared the application of the neutral differential equation in the formulation of mathematical models in many fields of engineering and sciences (see: [1,2,3,4]).

Neutral differential equation is a delay equation with time lag $r(t): E \to E$, $r(t) > 0$ incorporated in both the derivative and the state of the system. This time lag accounts for the non-instantaneous reaction of the system for any action. A general non homogeneous neutral differential equation is defined as

$$\frac{d^n}{dt^n}\rho(x(t), x(t-r)) = g(x(t)) + \eta(u(t)),\ n = 1,\ 2,\ \ldots, \tag{1.0}$$

where $g(x(t)),\ \eta(u(t))$ are the state and the input function of the system respectively.

The analysis of equation (1.0) begins with the establishment of the conditions for the existence and uniqueness of the system solution, which in most cases provides the methodological basis for computing the system solution. Many mathematical concepts have been used by researchers to achieve this aim (see: [3,5,6,7]). But in all the methods employed, the general idea is the establishment of boundeness and continuity of the functional in the space of its operation. The establishments of the uniqueness of the system solution also guaranties the analysis of the qualitative behavior of the solution for any perturbation of the system equations such as; asymptotic stability, controllability and observability. But these analyses are not easily come by, due to the transcendental character of the system equation. Considerable literature devoted to the study of the asymptotic and exponential stability behavior of solution of equation (1.0) abounds in [3,4,6,8,9,10]. Also, researches on the controllability are found in the works of ([1,11,12]).

The aim of this research work is to obtain a periodic solution of the linear neutral system of equation (1.0), with a constant delay $(r > 0)$ and an initial value $x(t_0) = \varphi_0$ by employing the variation of constant method. The inversion concept of a perturbed differential operator which yields the sum of a contraction and a compact map by Burton [5] is used as a tool in establishing the uniqueness of the periodic solution of the system. The exponential stability of the system equation is established by the analysis of the system resolvent matrix which must be negative definite. The resolvent matrix is also utilized to approximate the maximum delay bound for the system to be asymptotically stable. The controllability of the system is study by the analyses of the linear ordinary control and the free control parts of the system equations for; properness, non-singularity of the gramian matrix, canonical form of the controllable matrix and the non zero/ pole cancellation of the transfer function matrix. Application of the obtained results are employed on a neutral delay model of a partial element equivalent circuit (PEEC) consisting of a retarded mutual coupling between the partial inductance to confirm the suitability of the test.

2 Preliminary Results

Let $U \subset B_H([t-r,t], E),\ E \subset R^n$, where B_H is the Banach space of continuous functions and *and* $f \subset C_T^1(U)$, such that $f: E \times U \to R^n$ is a continuous mapping which is T- periodic in E and compact in U , then a first order time invariant neutral delay equation is defined as

$$\dot{x}(t) + \dot{x}(t-r) = f(t,, Ax(t) + Du(t)), \tag{2.0}$$

where $A \text{ and } D$ are $n \times n \text{ and } n \times 1$ matrices respectively, and $r > 0$ is the time lag.

The T-periodic solution is a vector function $x:[t-r,t] \to E$, which is dependence on the time lag $r > 0$ and the T -period in E such that

$$x(t+T) = x(t) \text{ and } f_{t+T} = f_t.$$

Considering the initial value linear system of (2.0) in the form

$$\frac{d}{dt}\big(x(t) - x(t-r)\big) = Ax(t) + Du(t)$$
$$x(t_0) = x_0, \tag{2.1}$$

with an output function $y(t) = Cx(t)$. By employing variation of constant method, (2.1) is expressed as

$$\int_{t-r}^{t} \dot{x}(\tau)e^{-A\tau}d\tau = x_0 + \int_{t_0}^{t} Du(\tau)e^{-A\tau}d\tau. \tag{2.2}$$

Since the solution vector function $x:[t-r,t] \to E$ is T-periodic, then (2.2) is equivalent to

$$\int_{t-T}^{t} \dot{x}(\tau)e^{-A\tau}d\tau = x_0 + \int_{t-r}^{t-T} \dot{x}(\tau)e^{-A\tau}d\tau + \int_{t_0}^{t} Du(\tau)e^{-A\tau}d\tau.$$

Integrating by part the term on the left and second term on the right yield

$$x(t)e^{-At} - x(t-T)e^{-A(t-T)} + A\int_{t-T}^{t} x(\tau)e^{-A\tau}d\tau =$$
$$x_0 - x(t-T)e^{-A(t-T)} - x(t-r)e^{-A(t-r)} + A\int_{t-T}^{t} x(\tau)e^{-A\tau}d\tau + \int_{t_0}^{t} Du(\tau)e^{A\tau}d\tau,$$
$$x(t) \quad = x_0e^{At} + x(t-r)e^{-A(t-r)} + A\Big[\int_{t-T}^{t} x(\tau)e^{-A(t-\tau)}d\tau + \int_{t-r}^{t-T} x(\tau)e^{-A(t-\tau)}d\tau \Big] + \int_{t_0}^{t} Du(\tau)e^{-A(t-\tau)}d\tau.$$

Re-arranging the above equation, the integral solution

$$x(t) = x_0e^{At} + x(t-r)e^{-A(t-r)} + A\int_{t-r}^{t} x(\tau)e^{-A(t-\tau)}d\tau + \int_{t_0}^{t} Du(\tau)e^{-A(t-\tau)}d\tau \tag{2.3}$$

is obtained.

3 Uniqueness of Solution (2.3)

The development of theory on uniqueness of solution (2.3) is enhanced by the utilization of some known results on the behavior of $f \text{ on } B_H$ stated as follows:

3.1 The Krasnoselskii Hypothesis

Krasnoselskii [13], states that the inversion of a perturbed differential operator yields the sum of a contraction and a compact map. This hypothesis is essential tool in the analysis of existence and uniqueness of solutions of the neutral differential equations as shown in [14,15]. The theorem combines the Banach contraction mapping principle and the Schauder fixed point theorem to establish bound conditions and convergent point in the non-empty closed convex subset of B_H. Burton [5] presents a modified form of the Krasnoselskii hypothesis as follows;

Theorem 3.1

Let $X \subset B_H$ be a close convex non empty subset, assume that Q and R map X into B_H such that $Q, R: X \to X$, then

i. for $x_1, x_2 \in X$, $Qx_1 + Rx_2 \in X$
ii. R is a contraction with contraction constant $0 \le k < 1$,
iii. Q is continuous and $Q(X)$ is contained in a compact set,

then there exists a unique $x \in X$ such that $Qx_1 + Rx_2 = x$.

Definitions 3.1

1. (X, ℓ) be a metric space and $R: X \to X$. R is a contraction such that for $\varphi, \phi \in X$ and $\varphi \neq \phi$, then $\ell(R\varphi, R\phi) < \ell(\varphi, \phi)$ and for $\varepsilon > 0$, *there exists* $0 \le k < 1$ such that $\varphi, \phi \in X$, $\ell(\varphi, \phi) \ge \varepsilon$ *implies that* $\ell(R\varphi, R\phi) \le k\, \ell\, f(\varphi, \phi)$
2. $X \subset B_H$ is a convex non empty subset with $\varphi, \phi \in X$, then X is a closed segment with boundary points φ, ϕ such that $\vartheta = \{v \in B_H / v = k\varphi + (1-k)\phi;\ 0 \le k \le 1\} \subset X$.
3. $X \subset B_H$ be a closed convex non empty subset and $\alpha: X \to X$ is a compact set in X if $\varphi(t) \in X$ there exist subsequent $\varphi(t_n)$, such that $[\alpha\varphi_i]_{i=0}^{n}$ converges in X

3.2 Mean value theorem

Theorem 3.2

Driver [16], states that if $f(x)$ is continuous and differentiable on the interval $[a,b]$, then there exists at least a number $c \in [a,b]$ such that

$$f(b) - f(a) = f'(c)(b-a) \quad (3.0)$$

3.3 Main Result

Theorem 3.3

Consider the general form of system (2.1) in the form of (1.0) as

$$\frac{d}{dt}\rho(x(t), x(t-r)) = f(x(t), u(t), t), \quad (3.1)$$

where $f(x(t),u(t),t) = g(x(t)) + \eta(u(t))$ is continuous and differential on $\Gamma = [t_0 - \alpha \le t \le t_0 + \alpha,\ \varphi(t_0) - \beta \le \varphi \le \varphi(t_0) + \beta]$, Lipschitzian in the t variable, and for constants $k_1, k_2 > 0,\ \|g(x(t))\| \le k_1,\ \|\eta(u(t))\| \le k_2$, then for any continuous initial value $\varphi : [t-h, t] \to E$ there exists a compact continuous periodic function $f_T(\varphi(t)) \in X \subset B_H$ which maps X to itself so that

$$f_T(\varphi(t)) = \varphi_0 e^{At} + \varphi_t(t) e^{-A(t-r)}) + A\int_{t-r}^{t} g_T(\varphi(\tau) e^{-A(t-\tau)}) d\tau + \int_{t_0}^{t} \eta_T(u(\tau) e^{-A(t-\tau)}) d\tau,$$

defined the unique T-periodic solution of (3.1)

Proof

The hypotheses of the theorem are proved in steps as follows:

Step (i); showing that $f(x(t),u(t),t)$ is Lipschitzian.

Assume $t_1, t_2 \in [t-r, t]$, then

$$\begin{aligned}\|f(x(t_2),u(t_2),t_2)) - f(x(t_1),u(t_1),t_1))\| &= \|[g(x(t_2)) - \eta(u(t_2))] - [g(x(t_1)) - \eta(u(t_1))]\| \\ &\le \|g(x(t_2)) - g(x(t_1))\| + \|\eta(u(t_2))) - \eta(u(t_1))\|\end{aligned}$$

By the mean value theorem,

$$\begin{aligned}\|[g(x(t_2)) - g(x(t_1))]\| &= g'(c)|t_2 - t_1| \\ &\le |t_1 - t_2|,\end{aligned}$$

and

$$\begin{aligned}\|\eta(u(t_2))) - \eta(u(t_1))\| &\le \eta'(c)|t_2 - t_1| \\ &\le |t_2 - t_1|, \qquad for \quad t_1 < c < t_2.\end{aligned}$$

Therefore

$$\begin{aligned}\|f(x(t_2),u(t_2),t_2)) - f(x(t_1),u(t_1),t_1))\| &\le \sup|t_2 - t_1| + \sup|t_2 - t_1| \\ &\le (k_1 + k_2)\rho(t_2,t_1) \\ &= M\rho(t_2,t_1),\end{aligned}$$

and hence $f(x(t),u(t),t)$ is Lipschitzian with Lipschitz constant $M > 0$.

Step (ii):

Consider the bounded closed rectangle

$$\Gamma^* = \{t,\ \varphi(s); t_0 - r \le s \le t,\ \varphi(t_0) - \beta \le \varphi(s) \le \varphi(t_0) + \beta, for\ 0 \le \beta < 1\}$$

such that $0 \le \beta k_1 < 1$ is satisfied. Also let $I = [t_0 - r,\ t]$ be a set, and $\bigcup(I) \subset X$ defined the space of all continuous functions $\omega(s)$ such that

$$\|\omega(s) - \varphi(t_0)\| \le \beta k_1 .$$

Then I is a continuous closed bounded subset of X and $\bigcup(I) \subset X$ is compact

Step (iii):

Using the hypothesis of Krasnoselskii as stated in [5], assume $Q, R : X \to X$, where Q is a contraction and R is continuous with the map $R\,X$ residing in a compact set I , then

$$f_T(\varphi(t)) = \varphi_0 e^{At} + \varphi_t(t)e^{-A(t-r)}) + A\int_{t-r}^{t} g_T(\varphi(\tau)e^{-A(t-\tau)})d\tau + \int_{t_0}^{t} \eta_T(u(\tau)e^{-A(t-\tau)})d\tau$$

can be express as sum of a contraction and a compact map. That is

$$f_T(\varphi(t)) = Q\varphi(t) + R\varphi(t) ,$$

with

$$Q(\varphi(t)) = \varphi_0 e^{At} + \int_{t_0}^{t} \eta_T(u(\tau)e^{-A(t-\tau)})d\tau$$

and

$$R(\varphi(t)) = \varphi_t(t)e^{-A(t-r)}) + A\int_{t-r}^{t} g_T(\varphi(\tau)e^{-A(t-\tau)})d\tau .$$

Step (iv):

Showing that $R\varphi(t)$ is compact in $X \subset B_H$. Consider the set $\bigcup(I) \subset X$ as defined in step (iii) with metric ℓ on $C[I_i]$, then for $R\varphi(s) \in X,\ s \in [t_0 - r,\ t]$

$$\|R\varphi(s) - \varphi(t_0)\| \le \beta .$$

Also for any continuous functions $\varphi(t_1),\ \varphi(t_2) \in X,\ \ for\ \ t_1, t_2 \in [t_0 - r, t]$, then

$$\ell(R\varphi(t_2), R\varphi(t_1)) = \sup\|R\varphi(t_2) - R\varphi(t_1)\|$$
$$\leq \beta\sup\|\varphi(t_2) - \varphi(t_1)\|$$
$$\leq \beta k_1$$
.

Thus, $R\varphi(t)$ is compact.

Step v:

Showing that $Q\varphi(t)$ map to itself

Consider any solution $\varphi(t) \in X \subset B_H$ and let Q be a map defined as $Q : \varphi(t) \to \varphi(t)$ such that $Q\varphi(t) = \varphi(t)$, where

$$\varphi(t) = \varphi_0 + \int_{t_0-r}^{t} \eta_T (u(\tau)e^{-A(t-\tau)})d\tau, \quad for \quad \varphi_0 = \varphi(t_0)e^{At_0}$$
.

Then

$$\|\varphi(t) - \varphi_0\| = \left\|\int_{t_0-r}^{t} \eta_T (u(\tau)e^{-A(t-\tau)})d\tau\right\|$$
$$\leq \int_{t_0-r}^{t} \|\eta_T (u(\tau)e^{-A(t-\tau)})\|d\tau$$
$$\leq \int_{t_0-r}^{t} k_2 d\tau = k_2|t - (t_0 - r)|$$
$$\leq k_2\alpha.$$

Showing that $Q\varphi(t) = \varphi(t)$ is a contraction: Assume for any arbitrary $\varphi(t_2)$,$\varphi(t_1) \in X \subset B_H$ with $Q\varphi(t_1) = \varphi(t_1)$, $Q\varphi(t_2) = \varphi(t_2)$, and for a metric ℓ on $C[I_i]$, then

$$\ell(Q\varphi(t_2), Q\varphi(t_1)) = \ell(\varphi(t_2), \varphi(t_1))$$
$$= \sup\left\|(\varphi_0 + \int_{t_0-r}^{t_2} \eta_T (u(\tau)e^{-A(t_2-\tau)})d\tau) - (\varphi_0 + \int_{t_0-r}^{t_1} \eta_T (u(\tau)e^{-A(t_2-\tau)})d\tau)\right\|$$
$$\leq \sup\int_{t_0-r}^{t} \left\|(\eta_{T_2} (u(\tau)e^{-A(t_2-\tau)}) - (\eta_{T_1} (u(\tau)e^{-A(t_2-\tau)})\right\|d\tau$$
$$\leq \sup\int_{t_0-r}^{t} \sup\|\eta_{T_2}(\tau) - \eta_{T_1}(\tau)\|d\tau$$
$$\leq k_2 \sup\int_{t_0-r}^{t} d\tau$$
$$\leq k_2 \sup|t - (t_0 - r)|$$
$$= k_2\alpha.$$

Thus, for $f_T(\varphi(t))$ satisfying conditions i – iv, then

$$f_T(\varphi(t)) = \varphi_0 e^{At} + \varphi_t(t)e^{-A(t-r)}) + A\int_{t-r}^{t} g_T(\varphi(\tau)e^{-A(t-\tau)})d\tau + \int_{t_0}^{t} \eta_T(u(\tau)e^{-A(t-\tau)})d\tau,$$

is the unique T-periodic solution of (3.1).

4 Exponential Asymptotic Stability

Considering the Laplace transform of (2.1) as

$$X(s) = \left(s - se^{-sr} - A\right)^{-1} x(0) + \left(s - se^{-sr} - A\right)^{-1} De^{-sr}, \tag{4.0}$$

such that

$$x(t) = L^{-1}\left(\left(s - se^{-sr} - A\right)^{-1} x(0) + \left(s - se^{-sr} - A\right)^{-1} De^{-sr}\right), \tag{4.1}$$

with resolvent matrix

$$\left(s - se^{-sr} - A\right)^{-1}, \tag{4.2}$$

and state transition matrix

$$\varphi(t) = L^{-1}\left(\left(s - se^{-sr} - A\right)^{-1}\right), \tag{4.3}$$

which mapped the initial state $x(0)$ to the state at time t and therefore defined the complementary solution

$$x(t) = \varphi(t)x(0). \tag{4.4}$$

By the definition of equation (4.3), the stability of system (2.1) largely depends on the resolvent matrix.

Definition 4.1

A matrix $A \in E^{n\times n}$ is called a Metzler matrix if all off-diagonal elements of $-A^{-1}$ are nonnegative (Ngoc, [10]).

Properties of Metzler matrix (Ngoc, [10]):

Suppose $A \in E^{n\times n}$ is a Metzler matrix,

i. Then $s(A)$ defined the eigenvalue of A
ii. There exists a non-negative eigenvector $x \neq 0$ such that $Ax = s(A)x$.
iii. Given any $\partial \in \Re$, there exists a nonzero vector $x \geq 0$ such that $Ax \geq \partial x$ if and only if $s(A) \geq \partial$

Definitions 4.2

i. System (2.1) is exponentially stable if for any initial value $x(0)=\varphi_0$, $\|\varphi(t)-\varphi_0\| \le \varepsilon$, for $\varphi(t)$ defined in equation (4.3)
ii. The zero solution of system (2.1) is exponentially asymptotically stable if for any initial value $x(0)=\varphi_0$, $\lim_{t\to\infty} \varphi(t)\,x(0)=0$, for $\varphi(t)$ defined as in equation (4.3)
iii. If the zero solution of (2.1) is exponentially asymptotically stable, then the system solution $x(t)$ of equation (4.1) is exponentially asymptotically stable

4.1 Stability Result

Theorem 4.1

Suppose $A \in E^{n\times n}$ is a Metzler matrix with $\|sI-A\| \ne 0$, such that the resolvent matrix in equation (4.2) is negative, then the zero solution of system (2.1) is exponentially asymptotically stable.

Proof

By the properties of Metzler matrix, if $-A^{-1} > 0$, then the geometric spectrum of A is the set

$$s_{\max} = \{s \in E,\ \partial \le s < 0,\ such\ \ that\ |sI-A| \ne 0\}. \tag{4.5}$$

Considering the resolvent matrix of equation (4.2), for any $\xi \in \Re$, such that $\xi I_n = s(I-e^{-sr})$, then

$$\left(s - se^{-sr} - A\right)^{-1} \cong \left(\xi I_n - A\right)^{-1} . \tag{4.6}$$

By Crammer's rule, $i\ j$ entries of equation (4.2)is written as

$$(-1)^{ij}\ \frac{\det \Lambda ij}{\det(\xi I - A)} = \frac{f_{ij}(\xi)}{X(\xi)} = P(\xi) , \tag{4.7}$$

where $f_{ij}(\xi)$ is a polynomial of degree less than n. By implication $P(\xi)$ defined the poles of equation (4.6), and so

$$P(\xi) \cong s_{\max} , \tag{4.8}$$

And therefore $\left(\xi I_n - A\right)^{-1}$ is stable.

Also by the binomial theorem;

$$\begin{aligned}\left(\xi I - A\right)^{-1} &= \left(\frac{I}{\xi}\right)\left(I - \frac{A}{\xi}\right)^{-1} \\ &= \frac{I}{\xi} + \frac{A}{\xi^2} + \frac{A^2}{\xi^3} + \ldots\end{aligned}$$

$$\varphi(t) = L^{-1}\left((\xi I - A)^{-1}\right) = I + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \ldots$$
$$= e^{At}.$$

By the definition (4.1) of the Metzler matrix and the approximation $s_{\max} \cong P(\xi)$, the resolvent matrix is negative definite, and

$$\lim_{t\to\infty}\varphi(t) = \lim_{t\to\infty} e^{At} = 0 .$$

Then the zero solution of system (2,1)is exponentially asymptotically stable, which implies exponential asymptotical stability of system (2.1).

4.2 Computation of the Delay bounds

In this section, results on the computation of the delay bounds of the system equations to be exponentially stable are presented. Chiasson [17] stated that the asymptotic stability of (2.1) is within certain range of the delay values. The Extension of [17], and utilizing the stability theory of section (4.2) (as largely dependent on the state transition matrix), the resolvent matrix is thus used as an approximating tool for evaluation of eigenvalues and the corresponding delay bounds for system (2.1) to be asymptotically stable. Hence

$$P(s, \delta(s)) = (s - s\delta - A)^{-1}, \tag{4.9}$$

which is a two variable polynomial with $\delta = e^{-sr}$. The auxiliary polynomial of equation (4.9) is

$$\bar{P}(s, \delta(s)) = -\delta^m P(s, \frac{1}{\delta}), \quad m = \deg\{P(s, \delta)\} .$$

Definition4.3 (Chiasson, [17])

Let $\{(s_i, \delta_i), i = 1, \ldots, k\}$ be the common zeros of $\left\{P(s, \delta), \bar{P}(s, \delta)\right\}$ for which $\operatorname{Re}(s_i) \le 0,\ s_i \ne 0$ and $|\delta_i| = 1,\ \delta_i \ne 1$. Then, for each such pairs (δ_i, s_i) , $r^* = \min_{r>0}\{r \in \Re / \delta_i = e^{-rs_i}\}$ defines the minimum bound of r_i

5 Controllability Result

The aim of this section is to develop theorems on the null controllability of system (2.0) for any initial condition $\varphi(t_0) = \varphi_0$, given a control function $u(t)$, by using the linear ordinary control system and the free control system of (2.1) . Hence Hermes and LaSalle [18] defined the associated linear ordinary control system of (2.0) as

$$\begin{aligned} \dot{x}(t) &= Ag(t, x(t)) + Du(t) \\ x(t_0) &= \varphi_0, \end{aligned} \tag{5.1}$$

and its corresponding integral trajectory as

$$f_T(t,t_0,\varphi_0,u) = \varphi_0 e^{At} + \int_{t_0}^{t-T} De^{A(t-\tau)}u(\tau)d\tau .$$

The condition for properness of (5.1) as stated in [18] is that, if the free system of (2.0) written as

$$\frac{d}{dt}L(t,x_t(t)) = Ag(t,x(t)), \tag{5.2}$$

for $L(t,x_t(t)) = x(t) - \rho(t,x(t-r))$, is uniformly asymptotically stable, then system (2.0) is null controllable.

Definition 5.1

i. **Proper system:** The system (5.1) is said to be proper on $[t_0,t_1]$ if for any vector $x \in E^{n\times 1}$, there exists a linear span of $De^{-A\tau}$ defined as $<x, De^{-A\tau}>=0$, $\tau \in [t_0,t_1]$, almost everywhere, an $rank[D : AD : A^2D \ldots : A^{n-1}D] = n$.
ii. **Complete controllability:** The system (5.1) is completely controllable on $[t_0,t_1]$ if for any initial function $\varphi(t_0) = \varphi_0$, there exists a control (input) function $u(t) \in E^{n\times 1}$, which is compact and can transfer the function to another state $\varphi(t_1) = \varphi_1$ in a finite time t_1.
iii. **Null controllability:** System (5.1) is null controllable on $[t_0,t_1]$, if for any $t_1 > t_0$, there is an admissible control function $u(\cdot):[t_0,t_1] \to E^{n\times 1}$ that transfer the periodic solution $f_T(t_0,\varphi_0,u,\tau) = x_0$ to $f_T(t_1,\varphi_1,u,\tau) = x_1 = 0$, for $\tau \in [t_0,t_1]$.
iv. **Reachable set:** The state of the system $x_1 \in E^n$, is reachable on $[t_0,t_1]$ if the exists an input function $u(\cdot):[t_0,t_1] \to E^{n\times 1}$ that transfer (x_0,t_0) to (x_1,t_1). The reachable set $R_t \subseteq E^n$ is the set of points reachable in t seconds such that

$$R_t = \int_{t_0}^{t_1} e^{A(t_1-\tau)}Du(\tau)d\tau . \tag{5.3}$$

Lemma 5.1

If the corresponding input trajectories of the linear control system of (5.1) takes values in the lager Hilbert space of integrable functions, such that $u(\cdot):[t_0,t_1] \to E^{n\times 1}$, for $t_1 \geq t_0$,then asymptotically proper system is controllable.

Proof:

Assume system (5.1) is asymptotically proper, using the result of [18], the integral trajectory of the linear ordinary control system is

$$\varphi_0 e^{At_1} + \int_{t_0}^{t_1} De^{A(t_1-\tau)}u(\tau)d\tau = 0,$$

so that

$$\varphi_0 = \int_{t_0}^{t_1} De^{-A\tau} u(\tau) d\tau.$$

Then, by definition (5.0-i), there exists a vector matrix $x \in E^n$ that span $De^{-A(t-\tau)}$ such that,

$$\left.\begin{aligned} &\int_{t_0}^{t} x^{T^*} De^{-A\tau} u(\tau) d\tau = 0, \\ &and \\ &x^{T^*} De^{-A\tau} u(\tau) = 0, \end{aligned}\right\} \tag{5.4}$$

hold almost everywhere.

By the properness assumption of lemma (5.1), if $u(t) \in (L([t_0, t_1], E^{n\times 1}), <\cdot,\cdot>)$, then equation (5.4) is satisfied if and only

$$<x^T, De^{-At}>= 0. \tag{5.5}$$

Defining

$$\chi(t) =< x, De^{-At} >= 0,$$

so that

$$\frac{d^{n-1}}{dt^{n-1}} \chi(t) = x^T [(A)^k De^{-At}] = 0, \; for \;\; k = 0,1,2...,(n-1) \tag{5.6}$$

Evaluating equation (5.6) at $t = 0, \; for \;\; k = 0,1,2,...n-1$, the controllability matrix is obtained as

$$\omega = [D : AD : \; A^2 D : \; A^3 D : \; ...: \; A^{n=1} D]. \tag{5.7}$$

Algebraically, system (5.6) has a solution if $< x, \omega >$ has n-linearly independent vector, hence

$$rank[D : AD : A^2 D : A^3 D : ...: A^{n-1} D] = n.$$

This implies controllability of system (2.1), hence the hypothesis of the Lemma (5.0) holds.

Definition 5.2

1 The reachability map on $[t_1, t_0]$ of the pair of matrices $(A(\cdot), B(\cdot))$ is the function

$$R_t : L([t_0, t_1], E^n) \to E^n$$
$$u \to \int_{t_0}^{t_1} e^{A(t_1-\tau)} Du(\tau)d\tau. \quad (5.8)$$

2 **Adjoint of a linear map:** Let $(U, F, <\cdot,\cdot>)$ and $(V, F, <\cdot,\cdot>)$ be Hilbert spaces and $H : U \to V$ is a continuous linear map, the adjoint of H is the linear map $H^* : V \to U$ defined by $< v, Du >=< D^* v, u >$, for all $u \in U, v \in V$.

3 **Self Adjoint:** Let $(U, F, <\cdot,\cdot>)$ and $H : U \to U$ be linear and continuous, H is called a self adjoint if and only if $H = H^*$, such that for $v, u \in U$ $< v, D(u) >=< D(v), u >$.

Therefore for the reachable set R_t of (5.3), assume $e^{A(t-\tau)}D = G(t)$, $t \in [t_0, t_1]$, and $x \in V$, then

$$< x, R_t > = < x, G(t)u > = x^T \left(\int_{t_0}^{t_1} G(t)u(t)dt \right)$$
$$= \int_{t_0}^{t_1} x^T G(t)u(t)dt$$
$$= \int_{t_0}^{t_1} (G(t)x)^T u(t)dt$$
$$= \int_{t_0}^{t_1} G(t)^T xu(t)dt$$
$$= < G(t)x, u >$$
$$= < R_t^*, x > .$$

Hence the linear map of the adjoint R_t^* is defined

$$R_t^* : E^n \to L([t_0, t_0], E^m)$$
$$\int_{t_0}^{t_1} e^{A(t_1-\tau)} Du(\tau)d\tau. \to u. \quad (5.9)$$

Lemma 5.2

The linear mapping $R_t o R_t^* : E^n \to E^n$ between two finite dimensional spaces admit a matrix representation known as the controllability gramian matrix

$$W(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t_1-\tau)} DD^T (e^{A(t_1-\tau)})^T d\tau \in E^{n\times n}$$

.

Proof:

Using equation (5.9), with definition (5.2), it is shown that $< x, R_t > = < R_t^*, x >$, and

$$R_t : u \to \int_{t_0}^{t_1} (e^{A(t_1-\tau)}) Du(\tau) d\tau$$
.

Let $[R_t(x)](\tau) = e^{A(t_1-\tau)} Dx$, then

$$[R_t o R_t^*](x) = \left[\int_{t_0}^{t_1} e^{A(t_1-\tau)} DD^T (e^{A(t_1-\tau)})^T d\tau \right] x$$
$$= W(t_0, t_1)x,$$

where $W(t_0, t_1)$ is the gramian matrix.

Theorem 5.1 (Main Result III)

Let $L : J \times E^n \times E^m \to E^n$ be well defined such that $L(t, x_t(t)) = x(t) - \rho(t, x(t-r))$, and system (2.1) is equivalent to

$$\begin{aligned} &\frac{d}{dt} L(t, x_t) = Ax(t) + Du(t) \\ &x(0) = \varphi_0 \\ &such\ that\ \ y(t) = Cx(t), \end{aligned} \tag{5.10}$$

with a transfer function defined as $C\left[sI - se^{-sr} - A\right]^{-1} D$. Then system (2.1) is null controllable if

i. The gramian matrix $W(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t-\tau)} DD^T e^{A(t-\tau)^T} d\tau$, is nonsingular,
ii. System (5.10) is asymptotically proper,
iii. The controllable matrix $\omega = [D : AD : \ldots : A^{n=1}D]$ has a canonical form,
iv. The transfer function has no zero/ pole cancellation.

Proof:

Consider the gramian matrix

$$W(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t-\tau)} DD^T e^{A(t-\tau)^T} d\tau$$
,

such that matrices $D = (d_{ij})\ and\ D^T = (b_{ij})$, for $b_{ij} = d_{ji}$. Then

$$|D| = \sum_{\delta \in S_n} (\text{sgn}\, \partial) d_{1\delta(1)} d_{2\delta(2)} \ldots d_{n\delta(n)}$$
,

and

$$\begin{aligned}\left|D^T\right| &= \sum_{\delta\in S_n}(\operatorname{sgn}\partial)b_{1\delta(1)}\,b_{2\delta(2)}\dots b_{n\delta(n)}\\ &= \sum_{\delta\in S_n}(\operatorname{sgn}\partial)d_{\delta(1)1}\,d_{\delta(2)2}\dots d_{\delta(n)n}\\ &= \sum_{\delta\in S_n}(\operatorname{sgn}\partial^{-1})d_{1\delta^{-1}(1)}\,d_{2\delta^{-1}(2)}\dots d_{n\delta^{-1}(n)},\end{aligned}$$

where S_n is the set of permutation, ∂ is the permutation and $\operatorname{sgn}\partial=\begin{cases}1, \textit{ if } \partial \textit{ is even}\\ -1, \textit{ if } \partial \textit{ is odd}\end{cases}$. Let $\vartheta = \partial^{-1}$ be the inverse permutation of ∂, and also $\operatorname{sgn}\partial^{-1} = \operatorname{sgn}\partial$, therefore $\operatorname{sgn}\partial^{-1} = \operatorname{sgn}\vartheta = \operatorname{sgn}\partial$. Hence,

$$\begin{aligned}\left|D^T\right| &= \sum_{\delta\in S_n}(\operatorname{sgn}\partial^{-1})d_{1\delta^{-1}(1)}\,d_{2\delta^{-1}(2)}\dots d_{n\delta^{-1}(n)}\\ &= \sum_{\delta\in S_n}(\operatorname{sgn}\vartheta)d_{1\Im(1)}\,d_{2\Im(2)}\dots d_{n\Im(n)}\\ &= |D|.\end{aligned}$$

Therefore by the definition of the gramian matrix,

$$W(t_0,t_1) = \int_{t_0}^{t_1} e^{A(t-\tau)}DD^T e^{A(t-\tau)^T}d\tau = \int_{t_0}^{t_1}\left\|e^{A(t-\tau)}D\right\|^2 d\tau \geq 0 \quad (5.11)$$

Since D is nonsingular by its properties, then

$$W(t_0,t_1) \neq 0,$$

and hypothesis (i) of the theorem is proved. Also by lemma (5.0) hypothesis (ii) is satisfied.

Proving (iii) using (ii); assume the controllable matrix $\omega = [D : AD :: \dots : A^{n=1}D]$ has C_k invariant cyclic subspace decomposition, with distinct eigenvalues, and then ω is diagonalizable with diagonal matrix

$$\omega = \begin{bmatrix} C_{m_1} & 0 & \dots & 0\\ 0 & C_{m_2} & \dots & 0\\ \vdots & & & \vdots\\ 0 & \dots & \dots & C_{m_k}\end{bmatrix},$$

where C_{m_k} defined the $\ker(\omega - s_k)$. Then, the characteristic polynomial of ω is,

$$\omega(t) = m_1(t) + m_2(t) + \dots m_k(t),$$

($m_k(t)$ is monic), and the minimal polynomial is,

$$m_\omega(t) = (t-s_1)^{m_1} + (t-s_2)^{m_2} + \ldots (t-s_k)^{m_r}.$$

Hence, the direct sum decomposition of $\Im$ is

$$\begin{aligned}\omega &= \ker(\omega - s_1)^{m_1} \oplus \ker(\omega - s_2)^{m_2}, \oplus \ldots \oplus (\omega - s_k)^{m_r} \\ &= C_1 \oplus C_2 \oplus \ldots \oplus C_k,\end{aligned}$$

where C_k defined the companion matrix. Defining a nilpotent operator $N^n = 0$, for $n \in Z^+$, with a minimum polynomial of index k written as $m_\omega(t) = (t-0)^{k_1} = t^k$ (clearly has eigenvalue zero), so that C_k can be express in terms of the nilpotent matrix as $C_k = s_k I_{rk} + N$ (I is an $r \times k$ identity matrix, N a nilpotent block) and the canonical form of the controllable matrix ω is thus

$$\omega = s_1 I_{r1} + N \oplus s_2 I_{r2} + N \oplus \ldots \oplus s_k I_{rk} + N. \tag{5.12}$$

Proving (iv); using the transform of equation (4.0) stated as

$$X(s) = \left(s - se^{-sr} - A\right)^{-1} x(0) + \left(s - se^{-sr} - A\right)^{-1} De^{-sr},$$

such that

$$x(t) = L^{-1}\left(\left(s - se^{-sr} - A\right)^{-1} x(0) + \left(s - se^{-sr} - A\right)^{-1} De^{-sr}\right),$$

and the output function

$$y(t) = L^{-1}\left(C\left(s - se^{-sr} - A\right)^{-1} x(0) + C\left(s - se^{-sr} - A\right)^{-1} De^{-sr}\right). \tag{5.13}$$

Then, there exists a mapping of the control function to the reacheable set defined as

$$\begin{aligned}\Im : L([0,t], E^n) &\to E^n \\ u &\to y,\end{aligned}$$

For

$$\Im_{yu}(s) = (sI - e^{-sr} - A)^{-1} De^{-sr}. \tag{5.14}$$

The function $\Im_{yu}(s)$ is the system transfer function of (5.10).

Algebraically, (5.14) is expressed as

$$\Im_{yu}(s) = \frac{b(s)}{a(s)} = K \frac{(s_1 - z_1) \ldots (s_i - z_i)}{(s_1 - p_1) \ldots (s_i - p_i)},$$

where poles (p_i) and zeros (z_i) are the roots of $a(s)$ *and* $b(s)$ respectively. Assume $z_i = p_i$, then

$$\lim_{z_i \to p_i} \Im_{yu} \to \infty,$$

which contradict the concept of controllability . Therefore $z_i \neq p_i$ (implies no zeros/poles cancellation).

6 Illustration

Consider the neutral delay model of a partial element equivalent circuit (PEEC) which includes new circuit element consisting of a retarded mutual coupling between the partial inductance of the form $L_p(t-r)$, but without retarded current sources of the form

$$\begin{aligned} &\frac{d^2}{dt^2}Q(t) - \frac{d^2}{dt^2}Q(t-r)) + a_1\frac{d}{dt}Q(t) + a_2Q(t) = Du(t) \\ &Q(t_0) = Q_0 \end{aligned} \tag{6.1}$$

with an output function $y(t) = CQ(t)$.

This is reduced to a first order system of the form

$$\begin{aligned} &\frac{d}{dt}(Q(t) - Q(t-r)) = AQ(t) + Du(t), \\ &Q(t_0) = Q_0, \\ &and \quad y(t) = CQ(t_0) \end{aligned} \quad , \tag{6.2}$$

where $A = \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ d_{21} \end{pmatrix}, \ C = (c_1 \ c_2)$.

6.1 Exponential Stability Analysis of system (6.2):

By theorem (4.1), A is a stable Metzler matrix ($s(A) < 0$), if $a_2 > \sqrt{a_2^2 - 4a_1}$ such that $-A^{-1} > 0$. Also, by the definition of the resolvent matrix

$$\begin{aligned} P(s,\delta(s)) = &\ \left(I(s - se^{-sr}) - A\right) \\ &= s^2(1 - 2\delta + \delta^2) + a_2 s(\delta - 1) + a_1 \\ \bar{P}(s,\delta(s)) &= -\delta^m P(s, \frac{1}{\delta(s)}) \\ &= -s^2(1 - 2\delta + \delta^2) - a_2 s(\delta^2 - \delta) - a_1, \end{aligned}$$

where m is the highest degree of $\delta \ in \ P(s,d(s))$, $\delta = e^{-sr}$. Solving $P(s,\delta(s)) \ and \ \bar{P}(s,\delta(s))$ simultaneously to obtained

$$s = -\frac{a_1}{a_2} \text{ and } \delta = \frac{-(a_1a_2^2 - 2a_1^2) \pm \sqrt{(a_1a_2^2 - 2a_1^2)^2 - 4a_1^2 92a_2^2 + a_1^2 - a_1a_2^2)}}{2a_1^2},$$

such that $\mathrm{Re}(s_i) \le 0$, $s_i \ne 0$ and $|\delta_i| = 1$, $\delta_i \ne 1$ are satisfied, then the system is asymptotically stable . Also, for each such pairs of (δ_i, s_i), $r^* = \min_{r>0} \{r \in \Re / \delta_i = e^{-rs_i}\}$ defines the minimum delay bounds of r_i for system (4.2) to be asymptotically stable.

6.2 Controllability Analysis of System (6.2) on $[t_0, t_1]$, with an Initial Function $Q(t_0) = Q_0$, Control (input) Function $Du(t) \in E^{m\times 1}$ and Output Function $y(t) = CQ(t)$ is as Follows:

1 By the proved of hypothesis (i) of theorem (5.1), the gramian matrix

$$W(t_0,t_1) = \int_{t_0}^{t_1} e^{A(t-\tau)} DD^T e^{A(t-\tau)^T} d\tau = \int_{t_0}^{t_1} e^{A(t-\tau)} D d\tau,$$

and

$$\exp A(t-\tau) = \exp\begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix}(t-\tau) = [v_1 v_2]\begin{pmatrix} e^{\lambda_1(t-\tau)} & 0 \\ 0 & e^{\lambda_2(t-\tau)} \end{pmatrix}[v_1 v_2]^{-1},$$

so that

$$W(t_0,t_1) = \int_{t_0}^{t_1} [v_1 v_2]\begin{pmatrix} e^{\lambda_1(t-\tau)} & 0 \\ 0 & e^{\lambda_2(t-\tau)} \end{pmatrix}[v_1 v_2]^{-1} D d\tau \ne 0,$$

where v_1, v_2 are the corresponding eigenvectors of the eigenvalues λ_1, λ_2 of matrix A , and hence $W(t_0, t_1)$ is nonsingular.

2 Analysis of hypothesis (ii) of theorem (5.1) for properness property ;

$$A = \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ d \end{pmatrix}, \quad AD = \begin{pmatrix} d \\ -a_2 d \end{pmatrix},$$

and

$$rank[D : AD] = rank\begin{pmatrix} 0 & d \\ d & -a_2 d \end{pmatrix}$$

$$= rank\begin{pmatrix} d & -a_2 d \\ 0 & d \end{pmatrix} = 2 \quad (by\, elementary\ rows\ \ operation).$$

Hence $[D : AD : \ldots : A^{n=1}D]$ is proper with rank 2.

3 Analysis of hypothesis (iii) of theorem (5.1) for the canonical form of controllable matrix $\omega = [D : AD] = \begin{pmatrix} 0 & d \\ d & -a_2 d \end{pmatrix}$ for the appropriate choice of matrices $A \text{ and } D$ is

$$\omega = s_1 I_2 + N_2 \oplus s_2 I_1 + N_1$$

where s_1, s_2 are the characteristic roots of the controllable matrix.

4 Analysis of hypothesis (iv) of theorem (5.1) for the transfer function;

$$C(s - s\delta - A)^{-1} D = \frac{\begin{pmatrix} c_1 & c_2 \end{pmatrix}\begin{pmatrix} s - s\delta - a_2 & 1 \\ -a_1 & s - s\delta \end{pmatrix}\begin{pmatrix} 0 \\ d \end{pmatrix}}{s^2(1 - 2\delta + \delta^2) + a_2 s(\delta - 1) + a_1}$$

$$= \frac{\begin{pmatrix} c_1(s - s\delta - a_2) - c_2 a_1 & c_1 + c_2(s - s\delta) \end{pmatrix}\begin{pmatrix} 0 \\ d \end{pmatrix}}{s^2(1 - 2\delta + \delta^2) + a_2 s(\delta - 1) + a_1}$$

$$= \frac{(c_1 + c_2(s - s\delta))d}{s^2(1 - 2\delta + \delta^2) + a_2 s(\delta - 1) + a_1},$$

which has no zero/pole cancellation?

Hence by analysis 1 – 4, and for the appropriate choice of matrices $A, D, and\ C$, system (6.2) is controllable.

7 Conclusion

The periodic solution of a linear neutral system with an input function and initial value was obtained by employing the variation of constant method. Theory on uniqueness of the obtained solution was established and proved by utilizing Burton [5] inversion theory of a perturbed differential operator which yields the sum of a contraction and a compact map. The resolvent matrix of the system equation (which must be negative definite) was used as a tool to analyze the exponential stability of the system equation and the computation of the maximum value of the delay bounds for the system to be asymptotically stable. The controllability of the system was studied by the analyses of the linear ordinary control and the free control parts of the linear neutral system:- for properness, non-singularity of the gramian matrix, canonical form of the controllable matrix and the non zero/ pole cancellation of the transfer function matrix. Results obtained were employed on neutral delay model of a partial element equivalent circuit (PEEC) consisting of a retarded mutual coupling between the partial inductance which confirmed the suitability of the test.

Competing Interests

Authors have declared that no competing interests exist.

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